Normal forms of ordinary linear differential equations in arbitrary characteristic

Florian Fürnsinn, Herwig Hauser

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Abstract

For every linear differential operator L in one variable with convergent or formal power series coefficients we construct a function space $\mathcal{F}$ on which L acts such that the equation $Ly = 0$ admits a basis of solutions in $\mathcal{F}$. In characteristic zero, this is done by adding to the ring $\mathcal{O}$ of holomorphic functions at 0 an abstract primitive z of $\frac{1}{x}$, say, the logarithm $\log(x)$. In positive characteristic p, the ring of constants consists of p-th powers and more primitives are required: namely, aside from $z_1 = z$ for $1/x$, also a primitive z_2 of z_1^{p-1}, and then, iteratively, a primitive z_{k+1} of z_k^{p-1}, is needed. The space $\mathcal{F}$ consists, in the case of positive characteristic, of formal power series in x whose coefficients are polynomials in countably many variables z_i.

It is then shown in all characteristics that the action of L on $\mathcal{F}$ possesses a *normal form*. It is given by the *initial operator* L_0 of L: The action of L on $\mathcal{F}$ is reduced to the action of L_0 by a linear automorphism u of $\mathcal{F}$, say, such that $L \circ u^{-1} = L_0$. As L_0 is an Euler operator, the equation $L_0 y = 0$ has the obvious solutions. From these, one obtains a full basis of solutions of $Ly = 0$ in $\mathcal{F}$ by pull-back with u^{-1}. This gives, in the holomorphic characteristic zero case, a concise formulation and proof of the theorems of Fuchs and Frobenius. As to positive characteristic, results of Dwork are extended, implications to Grothendieck's p-curvature conjecture are discussed, and the construction of the characteristic p exponential function is described.

1 Introduction

Let $L = p_n\partial^n + p_{n-1}\partial^{n-1} + \ldots + p_1\partial + p_0 \in \mathcal{O}[\partial]$ be a linear univariate differential operator with holomorphic or formal power series coefficients p_i in $\mathcal{O} = \mathbb{C}\{x\}$, respectively, $\mathcal{O} = \Bbbk[\![x]\!]$, $\Bbbk$ an arbitrary field. Write $L = \sum_{j=0}^{n}\sum_{i=0}^{\infty} c_{ij}x^i\partial^j$ for its expansion at 0, and denote by L_0 the *initial form* of L at 0, i.e., the Euler operator

$$L_0 = \sum_{i-j=\tau}^{\infty} c_{ij}x^i\partial^j,$$

where τ is the minimal *shift* $i - j$ occurring in the expansion. The *indicial polynomial* $\chi = \chi_L$ of L at 0 is defined as the polynomial $\chi(s) = \sum_{i-j=\tau} c_{ij}s^{\underline{j}}$ with $s^{\underline{j}} = s(s-1)\cdots(s-j+1)$, and its roots in $\mathbb{C}$, respectively in an algebraic closure $\overline{\Bbbk}$ of $\Bbbk$, are the *local exponents* of L at 0. Clearly, $L_0(x^k) = \chi(k)x^{k+\tau}$.

The objective of the present paper is to show that the operator L can be brought, by an automorphism u of a suitable function space $\mathcal{F}$ on which L acts, into the normal form L_0, when considered as a linear map on $\mathcal{F}$,

$$L \circ u^{-1} = L_0 : \mathcal{F} \to \mathcal{F}.$$

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In particular, the solutions $y(x)$ of the associated differential equation $L_0 y = 0$ in $\mathcal{F}$ give rise to solutions of $Ly = 0$ via $u^{-1}(y(x))$. If L_0 has the same order n as L - which corresponds to L having a *regular singularity* at 0 -, one thus recovers a basis of solutions of $Ly = 0$.

To formulate the respective normal form theorem with more detail, one has to distinguish the case of characteristic 0 from the case of positive characteristic $p > 0$.

Characteristic 0: We will equally consider holomorphic or formal power series coefficients, and write $\mathcal{O}$ for the $\Bbbk$-algebra of these, with $\Bbbk = \mathbb{C}$ or a field of zero characteristic. Denote by $\mathcal{K}$ the quotient field of $\mathcal{O}$, consisting of meromorphic functions, respectively formal Laurent series. It is well known that solutions of $Ly = 0$ may and most often will involve logarithms. We therefore extend $\mathcal{O}$ with the usual differentiation $\partial = \frac{d}{dx}$ to the differential ring $\mathcal{K}[z]$ with derivation ∂ defined by $\partial x = 1$ and $\partial z = \frac{1}{x}$. Here, the variable z plays the role of $\log(x)$ and is an abstract primitive of $\frac{1}{x}$. Accordingly, $L \in \mathcal{O}[\partial]$ induces a linear map on $\mathcal{K}[z]$, the *extension* of L, and again denoted by $L : \mathcal{K}[z] \to \mathcal{K}[z]$. If all shifts $i - j$ of L are ≥ 0, as we may and will assume upon multiplying L with a suitable monomial x^r, this map sends $\mathcal{O}[z]$ to $\mathcal{O}[z]$. This convention simplifies the notation.

Denote by $\Omega \subseteq \overline{\Bbbk}$ a (maximal) set of local exponents of L with *integer* differences. This makes sense since $\Bbbk$ has characteristic 0, and thus $\mathbb{Z} \subseteq \mathbb{Q} \subseteq \Bbbk$. We list the elements of Ω increasingly,

$$\rho_1 < \rho_2 < \cdots < \rho_r,$$

where $\rho_k < \rho_{k+1}$ stands for $\rho_{k+1} - \rho_k \in \mathbb{N}_{>0}$, and denote by $m_k \geq 1$ the respective multiplicity of ρ_k as a root of χ. Set $n_k = m_1 + \cdots + m_k$ and $n_0 = 0$. Then define the free $\mathcal{O}$-module

$$\mathcal{F}^\Omega = \sum_{k=1}^{r} \mathcal{O} x^{\rho_k}[z]_{<n_k} = \bigoplus_{k=1}^{r} \bigoplus_{i=n_{k-1}}^{n_k - 1} \mathcal{O} x^{\rho_k} z^i,$$

where x^ρ equals $\exp(\rho \log(x))$ if $\Bbbk = \mathbb{C}$, while, for arbitrary $\Bbbk$, it is just a symbol with derivation rule $\partial x^\rho = \rho x^{\rho-1}$. Any $L \in \mathcal{O}[\partial]$ with non-negative shifts acts naturally on $\mathcal{F}^\Omega$; we denote again by $L : \mathcal{F}^\Omega \to \mathcal{F}^\Omega$ the induced linear map.

Theorem 1.1 (see Theorem 2.15)**.** *Let $\mathcal{O}$ be the $\mathbb{C}$-algebra of holomorphic functions at 0, or, respectively, of formal power series. Let $L \in \mathcal{O}[\partial]$ be a linear differential operator with coefficients in $\mathcal{O}$ and shifts ≥ 0, and let L_0 denote its initial form. In the convergent setting, assume that 0 is a regular singularity of L, say* $\operatorname{ord} L_0 = \operatorname{ord} L$. *There exists a linear automorphism u of $\mathcal{F}^\Omega$ transforming the linear map L on $\mathcal{F}^\Omega$ into its initial form L_0, i.e., the following diagram commutes:*

$$\begin{tikzcd} \mathcal{F}^\Omega \arrow[d, "u"', "\wr"] \arrow[dr, "L"] & \\ \mathcal{F}^\Omega \arrow[r, "L_0"] & \mathcal{F}^\Omega \end{tikzcd}$$

A suitable automorphism u can be explicitly constructed from L. Varying the sets Ω of local exponents with integer differences one obtains:

Theorem 1.2 (see Theorem 2.16)**.** *Let $\mathcal{O}$, L, Ω and $\mathcal{F}^\Omega$ be as above. Assume that* $\operatorname{ord} L_0 = \operatorname{ord} L$. *Let x^ρ, $x^\rho z, \ldots, x^\rho z^{m_\rho - 1}$, for ρ of multiplicity m_ρ varying over the local exponents of L, be the canonical basis of solutions of $L_0 y = 0$ in $\mathcal{F} = \bigoplus_\Omega \mathcal{F}^\Omega$. Then*

$$u^{-1}(x^\rho), u^{-1}(x^\rho z), \ldots, u^{-1}(x^\rho z^{m_\rho - 1})$$

form a basis of solutions of $Ly = 0$ in $\mathcal{F}$.

Replacing z by $\log(x)$ one obtains in case $\mathcal{O} = \mathbb{C}\{x\}$ the classical theorem of Fuchs and Frobenius. Note that $\log(x)$ may appear in the solutions with powers up to the sum $n_k = m_1 + \cdots + m_k$ and not just up to the multiplicity m_k of ρ_k.

Positive Characteristic: Let now $\mathbb{k}$ be a field of characteristic $p > 0$, and let $\mathcal{O} = \mathbb{k}[\![x]\!]$ denote the ring of formal power series with quotient field $\mathcal{K} = \mathbb{k}(\!(x)\!)$. In this case, several complications arise: The derivation ∂ on $\mathcal{K}$ has $\mathcal{C} := \mathbb{k}(\!(x^p)\!)$ as field of constants, hence the linear independence of solutions of a differential equation $Ly = 0$ has to be taken over this field. As in characteristic zero we will need an abstract primitive of x^{-1}; it will be again denoted by z, taken as a variable, and satisfying $\partial z = \frac{1}{x}$. Note then that z^p will again be a constant, $\partial z^p = 0$. This implies that also z^{p-1} has no primitive in $\mathcal{O}[z]$. Now, when solving differential equations in characteristic p, one realizes that such a primitive is eventually needed: so one writes z_1 for z and introduces an extra variable z_2 with

$$\partial z_2 = \frac{1}{x} \cdot \frac{1}{z_1}.$$

Continuing in this way one is led to introduce a countable set of variables $z_1, z_2, \ldots$, abreviated by z, and related by the formal differentiation

$$\partial z_i = \frac{1}{x} \cdot \frac{1}{z_1 \cdots z_{i-1}}.$$

This formula mimics the differentiation rule for the i-fold composition $\log(\log(\ldots(\log(x))\ldots)$ of the complex logarithm. All this suggests to work over the field

$$\mathbb{k}(\!(x)\!)(z_1, z_2, \ldots) = \mathcal{K}(z)$$

of rational functions in z_i with formal Laurent series as coefficients. As it turns out, this field is still too small to solve differential equations in positive characteristic. One has to take instead the larger field

$$\mathbb{k}(z_1, z_2, \ldots)(\!(x)\!) = \mathbb{k}(z)(\!(x)\!).$$

Here, the coefficients of a monomial x^k may be rational functions whose numerators and denominators have arbitrarily large degree (this is not the case for $\mathcal{K}(z)$). Finally, one has to take care of monomials x^ρ where $\rho \in \overline{\mathbb{k}}$ is a local exponent of the operator. As ρ lies in an algebraic closure of $\mathbb{k}$, and thus $p \cdot \rho = 0$, the prospective module to be considered, namely,

$$\mathcal{F}^\rho := x^\rho \mathbb{k}(z)(\!(x)\!),$$

would not be well defined: for $k \in \mathbb{N}$, the product $x^\rho \cdot x^k$ could be equally read as $x^{\rho+p} \cdot x^{k-p} = x^\rho \cdot x^{k-p}$, with ambiguity in the second factor. To avoid this nuisance we introduce a further variable t "playing the role of a new x" and define

$$\mathcal{F}^\rho := t^\rho \mathbb{k}(z)(\!(x)\!)$$

together with the derivation $\partial t = \frac{1}{x} t$ as the relevant module. We will later define an even smaller subspace, restricting the powers of the variables z_i as coefficients of powers of x to obtain a more precise statement.

Theorem 1.3 (see Theorem 3.16)**.** *Let $\mathbb{k}$ be a field of positive characteristic p, and set $\mathcal{O} = \mathbb{k}[\![x]\!]$. Let $L \in \mathcal{O}[\partial]$ be a linear differential operator with coefficients in $\mathcal{O}$ and let L_0 denote its initial form. For every local exponent $\rho \in \overline{\mathbb{k}}$ of L, let $\mathcal{F}^\rho = t^\rho \mathbb{k}(z_1, z_2, \ldots)(\!(x)\!)$. There exists a linear automorphism u of $\mathcal{F}^\rho$ transforming the linear map L on $\mathcal{F}^\rho$ induced by L into its initial form L_0, i.e., such that*

$$L \circ u^{-1} = L_0 : \mathcal{F}^\rho \to \mathcal{F}^\rho.$$

We now pass on to the solutions of the associated differential equation $Ly = 0$. If $L \in \mathcal{O}[\partial]$ has initial form L_0, even constructing a basis of solutions of the "Euler equation" $L_0 y = 0$ is not obvious in positive characteristic. To do so, one has to specify first the field of constants in

$$\mathcal{R} = \bigoplus_{\rho \in \mathbb{k}} \mathcal{F}^\rho = \bigoplus_{\rho \in \mathbb{k}} t^\rho \mathbb{k}(z)(\!(x)\!).$$

Proposition 1.4 (see Proposition 3.3). *The field of constants of $\mathcal{R}$ is*

$$\mathcal{C} := \bigoplus_{\rho \in \mathbb{F}_p} t^\rho x^{p-\rho} \Bbbk(z^p)(\!(x^p)\!),$$

where $\mathbb{F}_p$ denotes the prime field of $\Bbbk$.

We then have

Proposition 1.5 (see Proposition 3.9). *Let $L_0 = \sum_{i-j=\tau} c_{ij} x^i \partial^j$ be an Euler operator. For any root ρ of χ_{L_0} in $\overline{\Bbbk}$, denote by m_ρ its multiplicity. A basis of solutions of $L_0 y = 0$ over the ring of constants $\mathcal{C}$ in $\mathcal{R}$ is given by*

$$y_{\rho,i} = t^\rho z^{i^*} = t^\rho z_1^i z_2^{\lfloor i/p^2 \rfloor} z_3^{\lfloor i/p^3 \rfloor} \cdots ,$$

where $i < m_\rho$ and $i^* = (i, \lfloor i/p \rfloor, \lfloor i/p^2 \rfloor, \lfloor i/p^3 \rfloor, \dots) \in \mathbb{Z}^{(\mathbb{N})}$ is a string of integers with finitely many non-zero entries. In particular, the dimension of the solution space of $L_0 y = 0$ in $\mathcal{R}$ over the constants is $n = \operatorname{ord} L_0$.

With this result in mind, the solutions of the general equation $Ly = 0$ go along the same line as in Theorem 1.2, using now Theorem 1.3 and Proposition 1.5.

Theorem 1.6 (see Theorem 3.17). *Let $\mathcal{O}$, L, ρ and $\mathcal{F}^\rho$ be as in Theorem 1.3. Assume that $\operatorname{ord} L_0 = \operatorname{ord} L$. Let $t^\rho, t^\rho z^{1^*}, \dots, t^\rho z^{(m_\rho - 1)^*}$, for ρ of multiplicity m_ρ varying over the local exponents of L, be the canonical basis of solutions of $L_0 y = 0$. Then*

$$u^{-1}(t^\rho), u^{-1}(t^\rho z^{1^*}), \dots, u^{-1}(t^\rho z^{(m_\rho - 1)^*})$$

form a basis of solutions over $\mathcal{C}$ of $Ly = 0$ in $\mathcal{R} = \bigoplus_\rho t^\rho \Bbbk(z)(\!(x)\!)$.

Example 1.7. The solution of the exponential differential equation $y' = y$ in characteristic 3, is given by

$$\begin{aligned} \exp_3 = {} & 1 + x + 2x^2 + 2x^3 z_1 + x^4(1 + 2z_1) + x^5 z_1 + 2x^6 z_1^2 + x^7(1 + 2z_1 + 2z_1^2) + x^8(2 + z_1^2) + \\ & + x^9(2z_1 + z_1^3 z_2) + x^{10}(2 + z_1 + 2z_1^2 + z_1^3 z_2) + \dots , \end{aligned}$$

see Example 4.1 for other characteristics.

Structure of the paper. Section 2.1 starts with a review of univariate differential operators, the definition of their initial form, the indicial polynomial and the local exponents. We describe the solutions of Euler equations and introduce the differentiation of a differential operator with respect to the exponents (in the sense of Frobenius). Then, in Section 2.2, we construct the function space $\mathcal{F}$ for equations in zero characteristic and then prove the respective normal form theorem, Theorem 2.15. This provides in Section 2.3 the description of a full basis of solutions in case the origin is a regular singular point, see Theorem 2.16. For irregular singularities, we sketch in Section 2.4 Merkl's algorithm of how to use the normal form theorem also in this case to obtain all solutions. The section also includes a brief discussion of the occurrence of apparent singularities and of Gevrey series in this context.

Chapters 3 and 4 are devoted to positive characteristic. We start with the construction of primitives, the enhanced enlargement of function spaces, and the respective ring of constants (Sections 3.1 and 3.2). These techniques are applied in section 3.3 for solving Euler equations in characteristic p. Section 3.4 contains the normal form theorem in positive characteristic, Theorem 3.16, together with its proof. This is then applied in section 3.5 to construct the associated solutions of differential equations with regular singularity (now defined through the order condition on the coefficients as given by Fuchs' criterion in characteristic 0), in Theorem 3.17.

With section 4.1 we begin to look at concrete examples as are the exponential function and the logarithm in characteristic p. In section 4.2 we study the case when only finitely many variables z_i

are needed to solve the equations, and relate this to the nilpotence of the p-curvature as described by Dwork. Also we ask and answer the question when the differential equation has even polynomial solutions, thus generalizing a result of Honda.

Section 4.3 compares the two normal form theorems with Grothendieck's p-curvature conjecture as well as with Bézivin's conjecture. The delicacy lies in the fact that the algorithm provided by the normal form theorem in positive characteristic is not the reduction modulo p of the characteristic 0 algorithm. The difference is subtle, and we aim at highlighting the involved phenomena (some of them being of purely number theoretic flavor). The article concludes in section 4.4 with the discussion of the integrality of the solutions, i.e., the question when the solutions of differential equations defined over $\mathbb{Z}$ have integer coefficients.

2 Differential equations in characteristic zero

2.1 Constructions with differential operators

Singular differential equations. Let be given a linear ordinary differential equation

$$Ly = p_n(x)y^{(n)} + p_{n-1}(x)y^{(n-1)} + \ldots + p_1(x)y' + p_0(x)y = 0,$$

where

$$L = p_n\partial^n + p_{n-1}\partial^{n-1} + \ldots + p_1\partial + p_0 \in \mathcal{O}[\partial]$$

is a differential operator. Here $\mathcal{O}$ denotes denotes the ring of germs of holomorphic functions in one variable x at a given chosen singular point of L, say, the origin 0, or the ring of polynomials $\Bbbk[x]$ or formal power series $\Bbbk[\![x]\!]$ over an arbitrary field $\Bbbk$ of any characteristic. Moreover, $\partial = \frac{d}{dx}$ denotes the usual derivative with respect to x. Writing $L = \sum_{j=0}^{n} \sum_{i=0}^{\infty} c_{ij}x^i\partial^j$, the operator decomposes into a sum

$$L = L_0 + L_1 + \ldots + L_m + \ldots$$

of *homogeneous* or *Euler operators* $L_k = \sum_{i-j=\tau_k} c_{ij}x^i\partial^j$, where the *shifts* $\tau_0 < \tau_1 < \ldots$ of the operators L_k are ordered increasingly and all L_k are assumed to be non-zero. The term L_0 of smallest shift constitutes the *initial form* of L at 0, and $\tau := \tau_0$ is called the shift of L at 0. Up to multiplying L with the monomial $x^{-\tau}$ we may assume (as we will do throughout) that L has shift $\tau = 0$; thus $L_0 = \sum_{i=0}^{n} c_{ii}x^i\partial^i$. The point $x = 0$ is singular for L if at least one quotient p_i/p_n has a pole at 0 (otherwise, 0 is called non-singular or ordinary). It is a *regular singularity* (in the sense of Fuchs) if L_0 has again order n, i.e., if $c_{nn} \neq 0$. The *indicial polynomial* of L at 0 is defined as

$$\chi_L(s) = \sum_{i=0}^{n} c_{ii}s^{\underline{i}} = \sum_{i=0}^{n} c_{ii}s(s-1)\cdots(s-i+1).$$

Here, $s^{\underline{i}}$ denotes the falling factorial or Pochhammer symbol. Clearly, $\chi_L = \chi_{L_0}$, which we simply denote by χ_0. Its roots ρ in the algebraic closure $\overline{\Bbbk}$ of $\Bbbk$ are the *local exponents* of L at 0, and $m_\rho \in \mathbb{N}$ will denote their multiplicity.

Remark 2.1. (i) We can rewrite any differential operator in terms of $\delta := x\partial$, the *Euler derivative.* The base change between $x^n\partial^n$ and δ is given by the Stirling numbers of the second kind $S_{n,k}$. This is readily verified using the recursion relation $S_{n+1,k} = kS_{n,k} + S_{n,k-1}$. This allows one to read off the indicial polynomial of an operator: If the initial form of an operator L is given by $L_0 = \varphi(\delta)$ for some polynomial φ, then the indicial polynomial of the operator is $\chi_L = \varphi$.

(ii) The classical characteristic zero definition of a regular singular point of a differential equation using the growth of the local solutions cannot be translated to characteristic p. However, the equivalent characterization by Fuchs using the order of vanishing of the coefficients of the equation applies.

We recall some basic facts from differential algebra. If (R, ∂) is a differential ring (or field), a *constant* is an element $r \in R$, such that $\partial r = 0$. The set of constants of R forms a subring (or subfield). A linear differential equation of order n has at most n linearly independent solutions in any differential field R over its field of constants. This is a simple corollary of Wronski's lemma, see [SP03], p. 9, or [Hon81]. A set of n linearly independent solutions is called a *full basis of solutions* of the equation in R. In particular, if $L \in \mathcal{O}[\partial]$ is a differential operator with holomorphic coefficients, then $Ly = 0$ can only have n $\mathbb{C}$-linearly independent solutions in $\mathcal{O}(\log(x))$.

From now on we stick to characteristic 0 and let $\mathcal{O}$ be the ring of germs of holomorphic functions at 0.

Euler equations. The solutions of Euler equations $L_0 y = 0$ are easy to find. They are of the form

$$y_{\rho,i} = x^\rho \log(x)^i,$$

where $\rho \in \mathbb{C}$ is a local exponent and i varies between 0 and $m_\rho - 1$. Here, $x^\rho = \exp(\rho \log(x))$ and $\log(x)$ may be considered either as a symbol subject to the differentiation rule $\partial x^\rho = \rho x^{\rho-1}$ and $\partial \log(x) = 1/x$, or as a holomorphic function on $\mathbb{C}_{\text{slit}} = \mathbb{C} \setminus \mathbb{R}_{\geq 0}$ or on arbitrary simply connected open subsets of $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

Extensions of differential operators. The consideration of logarithms is best formalized by introducing a new variable z for $\log(x)$ [Hon81], [Mez11]. To this end, equip the polynomial ring $\mathcal{K}[z]$ over the field $\mathcal{K} = \text{Quot}(\mathcal{O})$ of meromorphic functions at 0 with the $\mathbb{C}$-derivation

$$\begin{gathered}\partial : \mathcal{K}[z] \to \mathcal{K}[z],\\ \partial x = \partial x = 1, \qquad \partial z = x^{-1},\\ \partial(x^i z^k) = (iz + k) x^{i-1} z^{k-1}.\end{gathered}$$

This turns $\mathcal{K}[z]$ into a differential ring. It carries in addition the usual derivative ∂_z with respect to z. The same definition applies to $\mathcal{O}x^\rho[z]$ for any $\rho \in \mathbb{C}$, taking $\partial x^\rho = \rho x^{\rho-1}$.

Remark 2.2. In $\mathcal{K}[z]$ every element has a primitive; it is the smallest extension of $\mathcal{K}$ for which this holds true. Indeed, x^{-1} has no primitive in $\mathcal{K}$. The primitive of $x^{-1}z^\ell$ in $\mathcal{K}[z]$ is given by $\frac{1}{\ell+1} z^{\ell+1}$, while the primitive of $x^k z^\ell$ for $k \neq -1$ is given by $x^{k+1} p(z)$, where p is a polynomial of degree ℓ. Thus we may call $\mathcal{K}[z]$ the *primitive closure* of $\mathcal{K}$.

The j-fold composition $\partial \circ \cdots \circ \partial$ will be denoted by ∂^j. For a differential operator $L = p_n \partial^n + p_{n-1} \partial^{n-1} + \ldots + p_1 \partial + p_0 \in \mathcal{O}[\partial]$ define its *extension* as the induced action on $\mathcal{K}[z]$, denoted by the same letter,

$$L : \mathcal{K}[z] \to \mathcal{K}[z]$$

If $\rho \in \mathbb{C}$ is a local exponent of L, we will likewise associate to L the $\mathbb{C}$-linear map

$$L : \mathcal{K}x^\rho[z] \to \mathcal{K}x^\rho[z],\ x^\rho h(x) z^i \mapsto L(x^\rho h(x) z^i),$$

called again the extension of L to $\mathcal{K}x^\rho[z]$. Whenever L has shift $\tau \geq 0$ – as we will assume in the sequel – its extension sends $\mathcal{O}x^\rho[z]$ to $\mathcal{O}x^\rho[z]$ and thus defines a $\mathbb{C}$-linear map

$$L : \mathcal{O}x^\rho[z] \to \mathcal{O}x^\rho[z],\ x^\rho h(x) z^i \mapsto L(x^\rho h(x) z^i).$$

The Leibniz rule gives

Lemma 2.3. *Let L be an operator. Then, for $\rho \in \mathbb{C}$, $h \in \mathcal{O}$, and $i \geq 0$,*

$$L(x^\rho h(x) z^i)_{|z=\log(x)} = L(x^\rho h(x) \log(x)^i),$$

where on the right hand side L acts via $\frac{d}{dx}$. In particular, the map $\mathcal{O}x^\rho[z] \to \mathcal{O}x^\rho[\log(x)]$ given by the evaluation $z \mapsto \log(x)$ sends solutions of $Ly = 0$ to solutions of $Ly = 0$.

Example 2.4. The equation $x^2y'' + 3xy' + 1 = 0$ with Euler operator $L_0 = x^2\partial^2 + 3x\partial + 1$ has indicial polynomial $\chi_0 = \rho^{\underline{2}} + 3\rho^{\underline{1}} + 1 = (\rho+1)^2$ with double root $\rho = -1$. The solutions of $L_0 y = 0$ are $y_1 = x^{-1}$ and $y_2 = x^{-1}\log(x)$. The operator $L_0 = x^2\partial^2 + 3x\partial + 1$ on $\mathcal{O}x^{-1}[z]$ therefore has, as it should be, solutions x^{-1} and $x^{-1}z$. Indeed, $L_0(x^{-1}) = L_0(x^{-1}) = 0$, whereas $\partial(x^{-1}z) = x^{-2}(-z+1)$ and

$$\partial^2(x^{-1}z) = \partial(x^{-2}(-z+1)) = -2x^{-3}(-z+1) - x^{-3} = x^{-3}(2z-3)$$

give

$$L_0(x^{-1}z) = x^{-1}(2z-3) + 3x^{-1}(-z+1) + x^{-1}z = 0.$$

Function spaces. If L_0 is an Euler operator with exponents set $\Omega \subseteq \mathbb{C}$ and if m_ρ denotes the multiplicity of $\rho \in \Omega$, the $\mathbb{C}$-vector space

$$\mathcal{F}_0 = \sum_{\rho\in\Omega} \mathcal{O}x^\rho[z]_{<m_\rho}$$

of polynomials in z of degree $< m_\rho$ and with coefficients in $\mathcal{O}x^\rho$ is the correct space to look at for finding the solutions of the extended Euler equation $L_0 y = 0$, since these are of the form $x^\rho z^i$, for $\rho \in \Omega$ and $0 \le i < m_\rho$. The space $\mathcal{F}_0$ is, however, in general too small to contain the solutions of the extension $Ly = 0$ if $Ly = 0$ is a general equation with regular singularity and initial form L_0. A suitable enlargement of $\mathcal{F}_0$ is necessary. The method how to do this goes back to Fuchs, Frobenius, and Thomé; it requires some preparation.

Differentiating differential operators. This technique first appears in the works of Frobenius. If s is another variable, write the j-th derivative of $x^s = \exp(s\log(x))$ as $\partial^j x^s = s^{\underline{j}} x^{s-j}$. Define then, for $\ell \ge 1$, the *ℓ-th derivative* $(\partial^j)^{(\ell)}$ of ∂^j as

$$(\partial^j)^{(\ell)} x^s = (s^{\underline{j}})^{(\ell)} x^{s-j},$$

where $(s^{\underline{j}})^{(\ell)}$ denotes the ℓ-th derivative of $s^{\underline{j}}$ with respect to s. Clearly, $(\partial^j)^{(\ell)} = 0$ for $\ell > j$. Then, for a differential operator $L = p_n\partial^n + p_{n-1}\partial^{n-1} + \ldots + p_1\partial + p_0$ of order n, we get its ℓ-th derivative $L^{(\ell)}$ for $\ell \ge 1$ as

$$L^{(\ell)} = p_n \cdot (\partial^n)^{(\ell)} + p_{n-1} \cdot (\partial^{n-1})^{(\ell)} + \ldots + p_1 \cdot (\partial)^{(\ell)}.$$

This is no longer a differential operator; it is just a $\mathbb{C}$-linear map $\mathcal{O}t^\rho \to \mathcal{O}x^{\rho+\tau}$, where τ is the shift of L.

The following facts are readily verified, cf. Lemmata 3.6 and 3.7 for similar results in positive characteristic. Let L always be a differential operator of order n and shift $\tau \ge 0$. Let $\rho \in \mathbb{C}$ be arbitrary.

Lemma 2.5. *The extension of L to $\mathcal{O}x^\rho[z]$ has expansion*

$$L = L_x + L'_x\partial_z + \frac{1}{2!}L''_x\partial_z^2 + \ldots + \frac{1}{n!}L_x^{(n)}\partial_z^n,$$

where the $\mathbb{C}$-linear maps $L_x^{(\ell)}$ act on $\mathcal{O}x^\rho$ while leaving all z^i invariant, and ∂_z is the usual differentiation with respect to z.

Lemma 2.6. *If L_0 is an Euler operator of order n with shift 0, indicial polynomial $\chi_0(s)$, and extension L_0 to $\mathcal{O}x^\rho[z]$, then*

$$L_0(x^\rho z^i) = x^\rho \cdot [\chi_0(\rho)z^i + \chi_0'(\rho)iz^{i-1} + \frac{1}{2!}\chi_0''(\rho)i^{\underline{2}}z^{i-2} + \ldots + \frac{1}{n!}\chi_0^{(n)}(\rho)i^{\underline{n}}z^{i-n}].$$

Lemma 2.7. *The kernel of the extension L_0 to $\mathcal{F}_0 = \sum_{\rho\in\Omega}\mathcal{O}x^\rho[z]_{<m_\rho}$ of an Euler operator L_0 with exponents $\rho \in \Omega \subseteq \mathbb{C}$ of multiplicity m_ρ equals*

$$\mathrm{Ker}(L_0) = \bigoplus_{\rho\in\Omega}\bigoplus_{i=0}^{m_\rho-1} \mathbb{C}x^\rho z^i.$$

Lemma 2.8. *A $\mathbb{C}$-basis of solutions of an Euler equation $L_0 y = 0$ is given by*

$$x^\rho \log(x)^i,$$

where ρ ranges over all local exponents of L_0 at 0 and $0 \le i < m_\rho$, with m_ρ the multiplicity of ρ.

Example 2.9. (a) For the Euler operator $L_0 = x^2\partial^2 - 3x\partial + 3$ from before, with indicial polynomial $\chi_0(t) = (t+1)^2$ and exponent $\rho = -1$ of multiplicity $m_\rho = 2$, the extension $L_0 = x^2\partial^2 + 3x\partial + 1$ to $\mathcal{O}x^{-1}[z]$ has expansion

$$L_0(x^\rho z^i) = x^\rho[(\rho+1)^2 z^i + 2(\rho+1)iz^{i-1} + 2i(i-1)z^{i-2}]$$

and kernel

$$\mathrm{Ker}(L_0) = \mathbb{C}x^{-1} \oplus \mathbb{C}x^{-1}z.$$

(b) For the Euler operator $L_0 = x^3\partial^3 - 4x^2\partial^2 + 9x\partial - 9$ with indicial polynomial $\chi_0(t) = (t-1)(t-3)^2$ and exponents 1 and 3 of multiplicity one and two, respectively, the extension $L_0 = x^3\partial^3 - 4x^2\partial^2 + 9x\partial - 9$ to $\mathcal{O}x[z] \oplus \mathcal{O}x^3[z]$ has expansion

$$L_0(x^\rho z^i) = x^\rho[(\rho-1)(\rho-3)^2 z^i + (3\rho-5)(\rho-3)iz^{i-1} + (6\rho-14)i^{\underline{2}}z^{i-2} + 6i^{\underline{3}}z^{i-3}]$$

and kernel

$$\mathrm{Ker}(L_0) = \mathbb{C}x \oplus \mathbb{C}x^3 \oplus \mathbb{C}x^3 z.$$

Remark 2.10. In order to apply the perturbation lemma 2.14 below to an operator L acting on the space $\mathcal{F}_0 = \sum_{\rho\in\Omega} \mathcal{O}x^\rho[z]_{<m_\rho}$ one has to determine the image of the initial form L_0 of L. Write $L = L_0 - T$. Assuming that L_0 has shift 0, it follows that T is an operator with shift > 0, that is, it increases the order in x of elements of $\mathcal{F}_0$. Therefore, it sends $\mathcal{F}_0$ to $\mathcal{F}_0 x = \sum_{\rho\in\Omega} \mathcal{O}x^{\rho+1}[z]_{<m_\rho}$. One has no control about the precise image of T: it can be equal to whole $\mathcal{F}_0 x$ but it can also be much smaller. The perturbation lemma requires in any case the inclusion $\mathrm{Im}(T) \subseteq \mathrm{Im}(L_0)$ of images. This would trivially hold if L_0 were surjective onto $\mathcal{F}_0 x$. But this is not the case in general: it suffices to take $L_0 = x^2\partial^2 - x\partial$ with local exponents $\sigma = 0$ and $\rho = 2$, both of multiplicity one. Then $\mathcal{F}_0 = \mathcal{O} + \mathcal{O}x^2 = \mathcal{O}$ and $L_0 = L_0$. The image of $\mathcal{F}_0$ under L_0 is $L_0(\mathcal{F}_0) = \mathbb{C}x + \mathcal{O}x^3 \subsetneq \mathcal{O}x = \mathcal{F}_0 x$, with a gap at x^2. However, if $L = x^2\partial^2 - x\partial - x = L_0 - T$, the operator $T = x$ sends $x \in \mathcal{F}_0$ to $x^2 \notin L_0(\mathcal{F}_0)$. So the perturbation lemma does not apply to this situation. The way out of this dilemma is a further enlargement of $\mathcal{F}_0$ to a carefully chosen function space $\mathcal{F}$ containing $\mathcal{F}_0$. This enlargement will be explained in the next section.

2.2 The normal form of differential operators

When trying to lift, for an arbitrary operator L, the solutions $x^\rho \log(x)^k$ of $L_0 y = 0$ to solutions of $Ly = 0$, two obstructions occur. First, ρ might be a multiple root of the indicial polynomial and logarithms already appear in the solutions of $L_0 y = 0$. Second, if ρ is not a maximal exponent of L modulo $\mathbb{Z}$, that is, if $\rho + k$ is again an exponent of L for some $k > 0$, the lifting poses additional problems since higher powers of logarithms will occur among the solutions. We will approach and solve both problems simultaneously by using the extensions of operators L as defined above to appropriately chosen spaces $\mathcal{F}$ for which the image of the action of L_0 on $\mathcal{F}$ equals $\mathcal{F}x$. In this situation, the perturbation lemma 2.14 will apply to reduce $L : \mathcal{F} \to \mathcal{F}$ via a linear automorphism of $\mathcal{F}$ to L_0.

Enlargement of function spaces. As was done already classically [Fuc66] p. 136 and 157, [Fuc68], p. 362 and 364, [Tho72], p. 193, [Fro73], p. 221, it is appropriate to partition the set of exponents of a linear differential operator L into sets $\Omega \subseteq \mathbb{C}$ of exponents whose differences are integers and such that no exponent outside Ω has integer difference with an element of Ω. We list the elements of each Ω increasingly,

$$\rho_1 < \rho_2 < \cdots < \rho_r,$$

where $\rho_k < \rho_{k+1}$ stands for $\rho_{k+1} - \rho_k \in \mathbb{N}_{>0}$; denote by $m_k \geq 1$ the respective multiplicity of ρ_k as a root of the indicial polynomial χ_0 of L at 0. Set $n_k = m_1 + \cdots + m_k$ and $n_0 = 0$. To easen the notation, we omit in each ρ_k the reference to the respective set $\Omega = \{\rho_1, \dots, \rho_r\}$. Instead of $\mathcal{F}_0^\Omega = \sum_{k=1}^r \mathcal{O}x^{\rho_k}[z]_{<m_k}$ we will now allow polynomials in z of larger degree $< n_k$ and take the module

$$\mathcal{F}^\Omega = \sum_{k=1}^r \mathcal{O}x^{\rho_k}[z]_{<n_k} = \bigoplus_{k=1}^r \bigoplus_{i=n_{k-1}}^{n_k-1} \mathcal{O}x^{\rho_k}z^i = \bigoplus_{k=1}^{r-1} \bigoplus_{i=0}^{n_k-1} \bigoplus_{\sigma=\rho_k}^{\rho_{k+1}-1} \mathbb{C}x^\sigma z^i \oplus \bigoplus_{i=0}^{n_r-1} \mathcal{O}x^{\rho_r}z^i,$$

equipped with the derivation ∂ from before (see Figure 1). The two different direct sum decompositions of $\mathcal{F}$ will become relevant in a moment. Then set

$$\mathcal{F} = \bigoplus_\Omega \mathcal{F}^\Omega,$$

the sum varying over all sets Ω of exponents with integer difference. As each summand $\bigoplus_{i=n_{k-1}}^{n_k-1} \mathcal{O}x^{\rho_k}z^i$ of $\mathcal{F}^\Omega$ has rank m_k, it follows that $\mathcal{F}$ is free of rank n over $\mathcal{O}$.

Example 2.11. We illustrate the construction of the space $\mathcal{F}^\Omega$ with an example. Let

$$L = x^5\partial^5 - 2x^4\partial^4 - 2x^3\partial^3 + 16x^2\partial^2 - 16x\partial - x.$$

It has indicial polynomial $\chi(s) = s^2(s-2)(s-5)^2$. Therefore the local exponents are given by $\rho_1 = 0$, $\rho_2 = 2$ and $\rho_3 = 5$ with multiplicities $m_1 = 2$, $m_2 = 1$ and $m_3 = 2$. All local exponents differ by integers and we set $\Omega = \{0, 2, 5\}$ as well as $n_1 = 2$, $n_2 = 3$ and $n_5 = 5$. Then the space $\mathcal{F}^\Omega$ is given by

$$\mathcal{F}^\Omega = \mathcal{O} \oplus \mathcal{O}z \oplus \mathcal{O}x^2z^2 \oplus \mathcal{O}x^5z^3 \oplus \mathcal{O}x^5z^4.$$

The exponents (k, i) of monomials x^kz^i in $\mathcal{F}^\Omega$ are depicted in Figure 1.

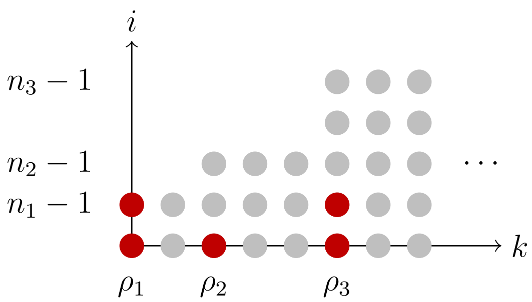

Figure 1: The sets of exponents (k, i) of monomials x^kz^i in $\mathcal{F}^\Omega$; in red monomials in $\mathrm{Ker}(L_0)$.

Example 2.12. This example will illustrate why local exponents with integer difference have to be treated in a separate and quite peculiar way. Assume that the Euler operator L_0 has just two local exponents σ and ρ of multiplicities m_σ and m_ρ, respectively, say $\Omega = \{\sigma, \rho\}$. If $\rho - \sigma \notin \mathbb{Z}$, then

$$\mathcal{F} = \mathcal{O}x^\sigma[z]_{<m_\sigma} \oplus \mathcal{O}x^\rho[z]_{<m_\rho};$$

if $\rho - \sigma \in \mathbb{N}$, then

$$\mathcal{F} = \mathcal{O}x^\sigma[z]_{<m_\sigma} + \mathcal{O}x^\rho[z]_{<m_\sigma+m_\rho} = \mathcal{O}x^\sigma[z]_{<m_\sigma} \oplus \mathcal{O}x^\rho z^{m_\sigma}[z]_{<m_\rho}.$$

The extension L_0 of L_0 to $\mathcal{F}$ has kernel $\mathbb{C}x^\sigma[z]_{<m_\sigma} \oplus \mathbb{C}x^\rho[z]_{<m_\rho}$ in the first case, and $\mathbb{C}x^\sigma[z]_{<m_\sigma} \oplus \mathbb{C}x^\rho z^{m_\sigma}[z]_{<m_\rho}$ in the second case. The respective images of L_0 are

$$\mathcal{O}x^{\sigma+1}[z]_{<m_\sigma} \oplus \mathcal{O}x^{\rho+1}[z]_{<m_\rho}$$

and

$$\mathcal{O}x^{\sigma+1}[z]_{<m_\sigma} \oplus \mathcal{O}x^{\rho+1}z^{m_\sigma}[z]_{<m_\rho},$$

so they equal $\mathcal{F}x$ in both cases.

If we would have taken in the second case where $\rho - \sigma \in \mathbb{N}$ is integral the space

$$\mathcal{F} = \mathcal{O}x^\sigma[z]_{<m_\sigma} + \mathcal{O}x^\rho[z]_{<m_\rho},$$

the image of L_0 would have been

$$\bigoplus_{k=1}^{\rho-\sigma-1} \mathbb{C}x^{\sigma+k}[z]_{<m_\sigma} \oplus \mathbb{C}x^\rho[z]_{<m_\sigma-m_\rho} \oplus \mathcal{O}x^{\rho+1}[z]_{<\max(m_\rho,m_\sigma)} \subsetneq \mathcal{F}x,$$

which is strictly included in $\mathcal{F}x$. Here $\mathbb{C}x^\rho[z]_{<m_\sigma-m_\rho}$ is to be read as 0 for $m_\sigma \leq m_\rho$. Indeed, $x^\rho z^{m_\sigma-1} \in \mathcal{F}x$ is not in the image of L_0. This would cause serious obstructions when trying to see L on $\mathcal{F}$ as a (negligible) perturbation of L_0, since the higher order terms of L may produce images in whole $\mathcal{F}x$. So the Perturbation Lemma 2.14 below would not apply.

The example suggests to admit in $\mathcal{F}$ powers of the logarithm, say, of z, which exceed the respective multiplicity of the local exponent ρ appearing in the factor x^ρ. The following lemma gives a precise answer of how to proceed; see Lemma 3.13 for the corresponding result in characteristic p.

Lemma 2.13. *Let $L \in \mathcal{O}[\partial]$ be an Euler operator with shift $\tau = 0$. Denote by $\Omega = \{\rho_1, \ldots, \rho_r\}$ a set of increasingly ordered local exponents ρ_k of L with integer differences and multiplicities m_k. Set $n_k = m_1 + \ldots + m_k$ and $\mathcal{F} = \mathcal{F}^\Omega = \sum_{k=1}^r \mathcal{O}x^{\rho_k}[z]_{<n_k}$.*

(a) The induced map $L : \mathcal{F} \to \mathcal{F}$ has image $\operatorname{Im}(L) = \mathcal{F}x = \sum_{k=1}^r \mathcal{O}x^{\rho_k+1}[z]_{<n_k}$.

(b) Its kernel $\operatorname{Ker}(L) = \bigoplus_{k=1}^r \mathbb{C}x^{\rho_k}[z]_{<m_k}$ *(cf. Lemma 2.7) has direct complement*

$$\mathcal{H} = \bigoplus_{k=2}^{r} \bigoplus_{i=m_k}^{n_k-1} \mathbb{C}x^{\rho_k}z^i \oplus \bigoplus_{k=1}^{r-1} \bigoplus_{e=1}^{\rho_{k+1}-\rho_k-1} \bigoplus_{i=0}^{n_k-1} \mathbb{C}x^{\rho_k+e}z^i \oplus \bigoplus_{i=0}^{n_r-1} \mathcal{O}x^{\rho_r+1}z^i,$$

in $\mathcal{F}$. Thus the restriction $L_{|\mathcal{H}}$ defines a linear isomorphism between $\mathcal{H}$ and $\mathcal{F}x$.

Proof. (a) We show first that L sends $\mathcal{F}$ into $\mathcal{F}x$. Recall from Lemma 2.6 that

$$L(x^\rho z^i) = x^\rho \cdot [\chi(\rho)z^i + \chi'(\rho)iz^{i-1} + \frac{1}{2!}\chi''(\rho)i^{\underline{2}}z^{i-2} + \ldots + \frac{1}{n!}\chi^{(n)}(\rho)i^{\underline{n}}z^{i-n}].$$

Therefore, as $\chi^{(\ell)}(\rho_k) = 0$ for $0 \leq \ell < m_k$, and using that $n_k - m_k = n_{k-1}$ for $k \geq 2$, it follows that L sends $\mathcal{F}$ into

$$\sum_{k=1}^r \mathcal{O}x^{\rho_k}[z]_{<n_k-m_k} = \sum_{k=2}^r \mathcal{O}x^{\rho_k}[z]_{<n_{k-1}} \subseteq \sum_{k=2}^r \mathcal{O}x^{\rho_{k-1}+1}[z]_{<n_{k-1}} \subseteq \mathcal{F}x.$$

Here, we use that $\rho_k - \rho_{k-1} \geq 1$. This proves $L(\mathcal{F}) \subseteq \mathcal{F}x$.

For the inverse inclusion $L(\mathcal{F}) \supseteq \mathcal{F}x$ it suffices to check that all monomials $x^\sigma z^i \in \mathcal{F}x$ lie in the image, where $\sigma = \rho_k + e$ for some $k = 1, \ldots, r$ and $e \geq 1$, and where $i < n_k$. We distinguish two cases.

(i) If $\sigma \notin \Omega$, proceed by induction on i. Let $i = 0$. By Lemma 2.5,

$$L(x^\sigma) = L_x(x^\sigma) + \sum_{j=1}^n \frac{1}{j!} L_x^{(j)} \partial_z^j(x^\sigma) = L_x(x^\sigma) = \chi(\sigma)x^\sigma \neq 0,$$

since σ is not a root of χ. So $x^\sigma \in L(\mathcal{F})$. Let now $i > 0$. Lemmata 2.5 and 2.6 yield

$$L(x^\sigma z^i) = L_x(x^\sigma z^i) + \sum_{j=1}^{n} \frac{1}{j!} L_x^{(j)} \partial_z^j (x^\sigma z^i) = \chi(\sigma) x^\sigma z^i + \chi^{(j)}(\sigma) x^\sigma \sum_{j=1}^{n} \frac{i^{\underline{j}}}{j!} z^{i-j}.$$

By the inductive hypothesis and using again that $\chi(\sigma) \neq 0$ we end up with $x^\sigma z^i \in L(\mathcal{F})$.

(ii) If $\sigma \in \Omega$, write $\sigma = \rho_k$ for some $1 \leq k \leq r$. As $x^\sigma z^i = x^{\rho_k} z^i \in \mathcal{F}x$ and $\rho_1 < \rho_2 < \cdots < \rho_r$, we know that $k \geq 2$ and

$$x^{\rho_k} z^i \notin x \cdot \sum_{\ell=k}^{r} \mathcal{O} x^{\rho_\ell} [z]_{<n_\ell}.$$

Hence

$$x^{\rho_k} z^i \in x \cdot \sum_{\ell=1}^{k-1} \mathcal{O} x^{\rho_\ell} [z]_{<n_\ell}.$$

This implies in particular that $0 \leq i < n_{k-1}$, which will be used later on. We proceed by induction on i. Let $i = 0$. By Lemma 2.5,

$$\begin{aligned}
L(x^{\rho_k} z^{m_k}) &= \sum_{j=0}^{m_k-1} \frac{1}{j!} L_x^{(j)} \partial_z^j (x^{\rho_k} z^{m_k}) + \frac{1}{m_k!} L_x^{(m_k)} \partial_z^{m_k} (x^{\rho_k} z^{m_k}) + \sum_{j=m_k+1}^{n} \frac{1}{j!} L_x^{(j)} \partial_z^j (x^{\rho_k} z^{m_k}) \\
&= \sum_{j=0}^{m_k-1} \frac{(m_k)^{\underline{j}}}{j!} \chi^{(j)}(\rho_k) x^{\rho_k} z^{m_k-j} + \chi^{(m_k)}(\rho_k) x^{\rho_k} \\
&= \chi^{(m_k)}(\rho_k) x^{\rho_k}.
\end{aligned}$$

Here, the sum in the first summand in the last but one line is 0 since ρ_k is a root of χ of multiplicity m_k, and for the same reason, the second summand $\chi^{(m_k)}(\rho_k) x^{\rho_k}$ is non-zero. So $x^\sigma = x^{\rho_k} \in L(\mathcal{F})$. Let now $i > 0$ and consider $x^\sigma z^i = x^{\rho_k} z^i \in \mathcal{F}x$. We will use that $i < n_{k-1}$ as observed above. Namely, this implies that $m_k + i < m_k + n_{k-1} = n_k$, so that $x^{\rho_k} z^{m_k+i}$ is an element of $\mathcal{F}$. Let us apply L to it. Similarly as in the case $i = 0$ we get

$$\begin{aligned}
L(x^{\rho_k} z^{m_k+i}) &= \sum_{j=0}^{m_k-1} \frac{1}{j!} L_x^{(j)} \partial_z^j (x^{\rho_k} z^{m_k+i}) + \frac{1}{m_k!} L_x^{(m_k)} \partial_z^{m_k} (x^{\rho_k} z^{m_k+i}) + \\
&\quad + \sum_{j=m_k+1}^{n} \frac{1}{j!} L_x^{(j)} \partial_z^j (x^{\rho_k} z^{m_k+i}) \\
&= \frac{(m_k+i)^{\underline{m_k}}}{m_k!} \chi^{(m_k)}(\rho_k) x^{\rho_k} z^i + \sum_{j=m_k+1}^{n} \frac{(m_k+i)^{\underline{j}}}{j!} \chi^{(j)}(\rho_k) x^{\rho_k} z^{m_k+i-j}.
\end{aligned}$$

The sum appearing in the second summand of the last line belongs to $L(\mathcal{F})$ by the induction hypothesis since $m_k + i - j < i$. As $\chi^{(m_k)}(\rho_k) \neq 0$, we end up with $x^\sigma z^i = x^{\rho_k} z^i \in L(\mathcal{F})$. This proves the inverse inclusion $L(\mathcal{F}) \supseteq \mathcal{F}x$ and hence assertion (a).

(b) From the shape of $\mathcal{F}$ and $\mathrm{Ker}(L)$ as depicted in Figure 1 one sees that $\mathcal{H}$ is a direct complement of $\mathrm{Ker}(L)$ in $\mathcal{F}$. Hence $L_{|\mathcal{H}}$ is automatically injective and $L(\mathcal{F}) = L(\mathcal{H}) = \mathcal{F}x$. $\square$

Here is the result from functional analysis required for the proof of the normal form theorems in characteristic 0 and p.

Lemma 2.14 (Perturbation Lemma). *If $\ell : F \to G$ is a continuous linear map between complete metric vector spaces which decomposes into $\ell = \ell_0 - t$ with $\mathrm{Im}(t) \subseteq \mathrm{Im}(\ell_0)$ and satisfies $|s(t(f))| \leq C \cdot |f|$, $0 < C < 1$, for a right inverse $s : \mathrm{Im}(\ell_0) \to F$ of $\ell_0 : F \to \mathrm{Im}(\ell_0)$ and all $f \in F$, then $u = \mathrm{Id}_F - st$ is a continuous linear automorphism of F which transforms ℓ into ℓ_0 via $\ell u^{-1} = \ell_0$.*

Proof. The prospective inverse of u is $v = \sum_{k=0}^{\infty}(st)^k$. It is well defined and continuous because of the estimate for $st(f)$ and the completeness of F. Hence u is an automorphism of F. From $\ell_0 s = \mathrm{Id}_{\mathrm{Im}(\ell_0)}$ it follows that $\ell_0 s \ell_0 = \ell_0$. From $\mathrm{Im}(t) \subseteq \mathrm{Im}(\ell_0)$ one gets that the compositions st and $s\ell$ are well defined and that $\ell_0 s \ell = \ell$ holds. Then

$$\begin{aligned}\ell_0 u &= \ell_0(\mathrm{Id}_F - st)\\ &= \ell_0(\mathrm{Id}_F - s(\ell_0 - \ell))\\ &= \ell_0(\mathrm{Id}_F - s\ell_0 + s\ell)\\ &= \ell_0 - \ell_0 s \ell_0 + \ell_0 s \ell\\ &= \ell_0 s \ell\\ &= \ell,\end{aligned}$$

as required. This proves the result. □

Theorem 2.15 (Normal form theorem in characteristic 0)**.** *Let $L \in \mathcal{O}[\partial]$ be a linear differential operator with holomorphic coefficients at 0, initial form L_0 and shift $\tau = 0$. Denote by $\Omega = \{\rho_1, \ldots, \rho_r\}$ a set of increasingly ordered local exponents ρ_k of L with integer differences and multiplicities m_k. Set $n_k = m_1 + \ldots + m_k$ and $\mathcal{F} = \mathcal{F}^{\Omega} = \sum_{k=1}^{r} \mathcal{O}x^{\rho_k}[z]_{<n_k}$. Let L, L_0 act on $\mathcal{F}$ via $\partial x = 1$ and $\partial z = x^{-1}$ as above. Assume that L has a regular singularity at 0.*

(a) The composition of the inverse $(L_{0|\mathcal{H}})^{-1} : \mathcal{F}x \to \mathcal{H}$ of $L_{0|\mathcal{H}}$ with the inclusion $\mathcal{H} \subseteq \mathcal{F}$ defines a right inverse $S_0 : \mathcal{F}x \to \mathcal{F}$ of L_0, again denoted by $(L_{0|\mathcal{H}})^{-1}$. Let $T : \mathcal{F} \to \mathcal{F}x$ be the extension of $T = L_0 - L$ to $\mathcal{F}$. The map

$$u = \mathrm{Id}_{\mathcal{F}} - S_0 \circ T : \mathcal{F} \to \mathcal{F}$$

is a linear automorphism of $\mathcal{F}$, with inverse $v = u^{-1} = \sum_{k=0}^{\infty}(S \circ T)^k : \mathcal{F} \to \mathcal{F}$.

(b) The automorphism v of $\mathcal{F}$ transforms L into L_0,

$$L \circ v = L_0.$$

(c) If 0 is an arbitrary (i.e., regular or irregular) singularity of L, statements (a) and (b) hold true with $\mathcal{O}$ replaced by the ring $\widehat{\mathcal{O}}$ of formal power series over $\mathbb{C}$ or over an arbitrary algebraically closed field K of characteristic 0.

Proof. Note first that u is well defined since the map T increases the order of series and thus sends $\mathcal{F}$ into $\mathcal{F}x$.

Once we show that $|S_0(T(f))| \le C|f|$ holds for some $0 < C < 1$ and all $f \in \mathcal{F}$, the perturbation lemma 2.14 implies that $u = \mathrm{Id}_{\mathcal{F}} - S_0 \circ T$ is a linear automorphism of $\mathcal{F}$ with $L \circ u^{-1} = L_0$, proving assertions (a) to (c) of the theorem. The proof of the estimate is split into two parts, first for formal power series and then for convergent ones, and uses a different metric in each case.

(i) Formal case: Denote by $\widehat{\mathcal{O}} = K[\![x]\!]$ the formal power series ring over an arbitrary field K of characteristic 0, equipped with the metric $d(f,g) = 2^{-\mathrm{ord}_0(f-g)}$, where ord_0 denotes the order of vanishing at 0. Let $\widehat{\mathcal{F}}$ denote the induced $\widehat{\mathcal{O}}$-modules $\widehat{\mathcal{F}} = \mathcal{F} \otimes_K \widehat{\mathcal{O}}$ and write again L for the extension $\widehat{L}$ to $\widehat{\mathcal{F}}$. As T increases the order of series in $\widehat{\mathcal{O}}$, while L_0 and S_0 do not decrease the order, it follows that also $S_0 \circ T$ increases the order. It thus satisfies the inequality $|S_0(T(f))| \le C \cdot |f|$ from the beginning, for some $0 < C < 1$, having set $|f| = d(f, 0) = 2^{-\mathrm{ord} f}$. Therefore the von Neumann series

$$v = \sum_{j=0}^{\infty}(S_0 \circ T)^j$$

defines a $\mathbb{C}$-linear map $v : \widehat{\mathcal{F}} \to \widehat{\mathcal{F}}$. Then it is clear that $v = u^{-1} = (\mathrm{Id}_{\widehat{\mathcal{F}}} - S \circ T)^{-1}$. So u and v are automorphisms, and $L \circ v = L_0$ by the perturbation lemma. This proves assertion (c) as well as the formal version of (a).

(ii) Convergent case: To prove the same thing inside $\mathcal{O}$, denote by $\mathcal{O}_s$ the subring of germs of holomorphic functions h such that $|h|_s < \infty$. Here, $s > 0$ and $|\sum_{k=0}^{\infty} a_k x^k|_s := \sum_{k=0}^{\infty} |a_k| s^k$. It is well known that the rings $\mathcal{O}_s$ are Banach spaces, and that $\mathcal{O} = \bigcup_{s>0} \mathcal{O}_s$ [GR71]. For $s > 0$ sufficiently small, u restricts to a linear map u_s on the induced Banach space $\mathcal{F}_s = (\sum_{k=1}^{r} \mathcal{O} x^{\rho_k}[z]_{<n_k})_s$. For the convergence of v_s it therefore suffices to prove that $||S_0 \circ T||_s < 1$, where $|| - ||_s$ denotes the operator norm of bounded linear maps $\mathcal{F}_s \to \mathcal{F}_s$. Once this is proven, $v_s = u_s^{-1}$ holds as before and shows that u_s and hence also u are linear isomorphisms. This argument provides a compact reformulation of Frobenius' proof for the convergence of solutions [Fro73], p. 218.

The inequality $||S_0 \circ T||_s < 1$ is equivalent to the existence of a constant $0 < C < 1$ such that

$$|S_0(T(x^\rho h(x) z^i))|_s \leq C \cdot |x^\rho h(x) z^i|_s$$

for all $x^\rho h(x) z^i \in \mathcal{F}_s$. This will imply in particular that $(S_0 \circ T)(\mathcal{F}_s) \subseteq \mathcal{F}_s$.

We will treat the case where ρ is a maximal local exponent of L modulo $\mathbb{Z}$ and a simple root of χ_0. In this case, no extensions of operators are required, and we can work directly with L, S and T and $\mathcal{F} = \mathcal{O} x^\rho$. For non-maximal exponents there occur notational complications which present, however, no substantially new difficulty. So we shall omit the general case. For $h = \sum_{k=0}^{\infty} a_k x^k \in \mathcal{O}$ and writing $L = \sum_{j=0}^{n} p_j(x) \partial^j$ with $p_j = \sum_{i=0}^{\infty} c_{ij} x^i$ we have

$$T(x^\rho h) = - \sum_{i-j>0} \sum_{k=0}^{\infty} (\rho + k)^{\underline{j}} c_{ij} a_k x^{\rho+k+i-j},$$

and, recalling that L_0 is assumed to have shift 0,

$$S(T(x^\rho h)) = - \sum_{i-j>0} \sum_{k=0}^{\infty} \frac{(\rho + k)^{\underline{j}}}{\chi_L(\rho + k + i - j)} c_{ij} a_k x^{\rho+k+i-j}.$$

As $i - j > 0$, $k \geq 0$, and ρ is maximal, no $\rho + k + i - j$ appearing in the denominator is a root of χ_L. Hence the ratio

$$\frac{(\rho + k)^{\underline{j}}}{\chi_L(\rho + k + i - j)} = \frac{(\rho + k)^{\underline{j}}}{\sum_{\ell=0}^{n} c_{\ell,\ell} (\rho + k + i - j)^{\underline{\ell}}}$$

is well defined. But $c_{n,n} \neq 0$ since 0 is a regular singularity of L, and hence $(\rho + k + i - j)^{\underline{n}}$ appears in the denominator with non-zero coefficient. As $j \leq n$ this ensures that the ratio remains bounded in absolute value, say $\leq c$, as k tends to ∞. Taking norms on both sides of the above expression for $S(T(x^\rho h))$ yields, for $s \leq 1$ and $h \in \mathcal{O}_s$, the estimate

$$|S(T(x^\rho h))|_s \leq c \sum_{i-j>0} \sum_{k=0}^{\infty} |c_{ij}||a_k| s^{\rho+k+i-j} = c \sum_{i-j>0} |c_{ij}| s^{i-j} \sum_{k=0}^{\infty} |a_k| s^{\rho+k}$$

But by assumption, $p_j = \sum_{i=0}^{\infty} c_{ij} x^i \in \mathcal{O}_s$ for all $0 < s \leq s_0$ and all $j = 0, \ldots, n$. This implies in particular $\sum_{i>j}^{\infty} c_{ij} x^i \in \mathcal{O}_s$ and then, after division by x^{j+1} and since $i > j$, that

$$\sum_{i>j}^{\infty} c_{ij} x^{i-(j+1)} \in \mathcal{O}_s.$$

We get for all $s \leq r$ that

$$\sum_{i-j>0} |c_{ij}| s^{i-j} = s \cdot \sum_{i-j>0} |c_{ij}| s^{i-(j+1)} \leq c' s$$

for some $c' > 0$ independent of s. This inequality allows us to bound $|S(T(x^\rho h))|_s$ from above by

$$|S(T(x^\rho h))|_s \leq cc's \sum_{k=0}^{\infty} |a_k| s^{\rho+k} = cc's|x^\rho h|_s.$$

Take now $s_0 > 0$ sufficiently small, say $s_0 \leq \min(1, r)$ and $s_0 < \frac{1}{cc'}$, and get a constant $0 < C < 1$ such that for $0 < s \leq s_0$ one has

$$|S(T(x^\rho h))|_s \leq C \cdot |x^\rho h|_s.$$

This establishes $||S \circ T||_s < 1$ on $\mathcal{F}_s$ for $0 < s \leq s_0$. By the Perturbation Lemma 2.14, $u_s = \mathrm{Id}_{\mathcal{F}_s} - S \circ T$ is an automorphism of $\mathcal{F}_s$ with inverse $v_s = \sum_k (S \circ T)^k$. This completes the proof of the theorem. □

2.3 Solutions of regular singular equations

As a first consequence of the normal form theorem 2.15 we recover the classical theorem of Fuchs from 1866 and 1868 about the local solutions of differential equations at regular singular points [Fuc66], [Fuc68]. The statement was reorganized and further detailed by Thomé and Frobenius in a series of papers between 1872 and 1875 [Tho72], [Tho73a], [Tho73b], [Fro73], [Fro75]. See also [Fab85] formula (9), p. 19.

Theorem 2.16 (Local solutions in characteristic 0)**.** *Let $L \in \mathcal{O}[\partial]$ be a linear differential operator with holomorphic coefficients and regular singularity at 0. For each set Ω of local exponents of L with integer differences, let $u_\Omega : \mathcal{F}^\Omega \to \mathcal{F}^\Omega$ be the automorphism of assertion (a) of the normal form theorem.*

(a) Varying Ω, a $\mathbb{C}$-basis of local solutions of $Ly = 0$ at 0 is given by

$$y_{\rho,i}(x) = u_\Omega^{-1}(x^\rho z^i)_{|z=\log(x)},$$

for $\rho \in \Omega$ a local exponent of L of multiplicity m_ρ, and $0 \leq i < m_\rho$.

(b) Order the exponents in a chosen set Ω as $\rho_1 < \ldots < \rho_r$ and write m_k for m_{ρ_k}. Set $n_k = m_1 + \ldots + m_k$. Each solution related to Ω is of the form, for $1 \leq k \leq r$ and $0 \leq i < m_k$,

$$y_{\rho_k,i}(x) = x^{\rho_k}[f_{k,i} + \ldots + f_{k,0}\log(x)^i] + \sum_{\ell=k+1}^{r} x^{\rho_\ell} \sum_{j=n_{\ell-1}}^{n_\ell - 1} h_{k,i,j}(x)\log(x)^j,$$

with holomorphic $f_{k,i}$ and $h_{k,i,j}$ in $\mathcal{O}$, where $f_{k,0}$ has non-zero constant term.

Proof. Let Ω be a set of local exponents of L at 0 with integer differences and consider the space $\mathcal{F}^\Omega = \sum_{k=1}^{r} \mathcal{O}x^{\rho_k}[z]_{<n_k}$ as in the statement of the normal form theorem. Extend L and L_0 to $\mathcal{F} = \bigoplus_\Omega \mathcal{F}^\Omega$. By Lemma 2.7, a $\mathbb{C}$-basis of solutions of L_0 is given by the monomials $x^\rho z^i$, $0 \leq i \leq m_\rho - 1$, where ρ is a local exponent of multiplicity m_ρ. By assertion (d) of the normal form theorem and since L and L_0 have the same order n, the pull-backs $u^{-1}(x^\rho z^i)$ form a $\mathbb{C}$-basis of solutions of $Ly = 0$. Now Lemma 2.3 gives the result. □

Remark 2.17. The coefficient functions $f_{k,i}$ and $h_{\rho,i,j} \in \mathcal{O}$ of the solutions in assertion (b) of the theorem are related to each other. For instance, if ρ is a maximal exponent in Ω of multiplicity m_ρ, then

$$\begin{aligned}
y_{\rho,0} &= x^\rho \cdot g_0 \\
y_{\rho,1} &= x^\rho \cdot [g_1 + g_0 \log(x)] \\
&\cdots \\
y_{\rho,m_\rho-1} &= x^\rho \cdot [g_{m_\rho-1} + g_{m_\rho-2}\log(x) + \ldots + g_1 \log(x)^{m_\rho-2} + g_0 \log(x)^{m_\rho-1}],
\end{aligned}$$

with holomorphic $g_0, \ldots, g_{m_\rho-1}$, where g_0 has non-zero constant term.

Example 2.18. (i) If L has exactly two exponents σ and ρ, with $\rho - \sigma \in \mathbb{N}_{>0}$ and of multiplicities m_σ and m_ρ, respectively, we get accordingly

$$\mathcal{F} = x^\sigma[\mathcal{O} \oplus \cdots \oplus \mathcal{O}z^{m_\sigma - 1}] + x^\rho[\mathcal{O} \oplus \cdots \oplus \mathcal{O}z^{m_\sigma + m_\rho - 1}].$$

which we rewrite as

$$\mathcal{F} = x^\sigma[\mathcal{O} \oplus \cdots \oplus \mathcal{O}z^{m_\sigma - 1}] \oplus x^\rho[\mathcal{O}z^{m_\sigma} \oplus \cdots \oplus \mathcal{O}z^{m_\sigma + m_\rho - 1}].$$

A basis of solutions of $Ly = 0$ are $\mathcal{O}$-linear combinations

$$\begin{aligned}
y_{\sigma,0} &= x^\sigma \cdot h_0 + x^\rho g_0 \log(x)^{m_\sigma} \\
y_{\sigma,1} &= x^\sigma \cdot [h_1 + h_0 \log(x)] + x^\rho \log(x)^{m_\sigma}[g_1 + g_0 \log(x)] \\
&\dots \\
y_{\sigma,m_\sigma-1} &= x^\sigma \cdot [h_{m_\sigma-1} + h_{m_\sigma-2} \log(x) + \cdots + h_1 \log(x)^{m_\sigma-2} + h_0 \log(x)^{m_\sigma-1}] + \\
&\quad + x^\rho \log(x)^{m_\sigma} \cdot [g_{m_\rho-1} + \cdots + g_0 \log(x)^{m_\rho-1}] \\
y_{\rho,0} &= x^\rho \cdot f_0 \\
y_{\rho,1} &= x^\rho \cdot [f_1 + f_0 \log(x)] \\
&\dots \\
y_{\rho,m_\rho-1} &= x^\rho \cdot [f_{m_\rho-1} + f_{m_\rho-2} \log(x) + \ldots + f_1 \log(x)^{m_\rho-2} + f_0 \log(x)^{m_\rho-1}]
\end{aligned}$$

with holomorphic $f_0, \ldots, f_{m_\rho-1}$, $g_0, \ldots, g_{m_\rho-1}$, $h_0, \ldots, h_{m_\sigma-1}$.

(ii) The function $e^x \log(x)$ satisfies the differential equation $Ly = 0$ for

$$L = x^2\partial^2 + (1 - 2x)x\partial + x(x - 1).$$

A basis of solutions is completed by e^x. The initial form of L is

$$L_0 = x^2\partial^2 + x\partial.$$

Consequently, the only local exponent of L is 0 with multiplicity 2. The basis of solution is, as expected, contained in

$$\mathcal{O} \oplus \mathcal{O}z.$$

2.4 Applications in characteristic zero

Irregular singularities. Whenever the point 0 is an irregular singularity of a differential operator $L \in \mathcal{O}[\partial]$ with holomorphic coefficients, i.e., when $n_0 = \operatorname{ord} L_0 < \operatorname{ord} L = n$, Theorem 2.16 does not provide a basis of solutions of $Ly = 0$, but only n_0 linearly independent solutions thereof. It is well known that the solutions which are missing for a full basis are more complicated and may have essential singularities [Fab85]. More specifically, they are of the form

$$y(x) = \exp(q(x)) \cdot x^\rho \cdot \Big[h_0(x) + h_1(x)\log(x) + \ldots + h_k(x)\log^k(x)\Big],$$

where $q \in \mathbb{C}(x)$ is a rational function, $\rho \in \mathbb{C}$ a local exponent of L, and h_i holomorphic [Sal19], Thm. 3, [Inc44], Chap. XVII, p. 417. Actually, one can even take for q a Laurent polynomial

$$q(x) = \sum_{r \in \mathbb{Q}_{>0}} c_r x^{-r},$$

with $c_r \in \mathbb{C}$, almost all $c_r = 0$. It suffices to take here $r > 0$ since summands with non-negative exponents produce holomorphic factors in $y(x)$. In [Mer22], Nicholas Merkl describes an algorithm how to construct these solutions by reducing the differential equation $Ly = 0$ to various differential equations $\widetilde{L}y = 0$, all with regular singularity at 0, to apply then to these new equations the normal form theorem in characteristic 0, Theorem 2.15, to obtain their respective solutions as in 2.16. It then suffices to pull back these solutions to the original equation via the inverse conjugations. Doing this for all induced equations $\widetilde{L}y = 0$, one eventually obtains a basis of solutions of $Ly = 0$.

This shows that the normal form theorem 2.15 is applicable to *all* linear differential equations with holomorphic coefficients to construct their solutions. In the irregular case, there is also a method to find the solutions using the *Newton polygon* of L: it is similar in substance, though more computational, see [SP03], section 3.3, p. 90.

We briefly sketch Merkl's algorithm: Let be given an operator $L \in \mathcal{O}[\partial]$ of order n. Denote by $\delta = x\partial$ the basic Euler operator, and define, for $r \in \mathbb{Q}_{\geq 0}$ a positive rational number, the weighted operator

$$\delta_r = x^r \delta.$$

Here, x^r is considered either as a symbol or as a *Puiseux monomial* with $(x^r)' = rx^{r-1}$. Writing $r = e/d$ with $e, d \in \mathbb{N}$, we may then expand formally L as a linear combination

$$L = \sum_{j=0}^{n} \sum_{i=0}^{\infty} c_{ij} x^{i/d} \delta_r^j,$$

with coefficients $c_{ij} x^{i/d}$. For each j, let $i = i_j \in \mathbb{N}$ be minimal with $c_{ij} \neq 0$ (we suppress here the reference to r). Then define the *weighted initial form* $L_{0,r}$ and the *weighted indicial polynomial* χ_r of L with respect to r as

$$L_{0,r} = \sum_{j=0}^{n} c_{i_j j} \delta_r^j \in \mathbb{C}[\delta_r],$$

$$\chi_r = \sum_{j=0}^{n} c_{i_j j} s^j \in \mathbb{C}[s].$$

For $r = 0$ we just get the classical initial form $L_0 = L_{0,0}$ and its indicial polynomial $\chi = \chi_0$ defined earlier. Note that for generic r, the polynomial χ_r will be a monomial and hence have the unique root 0 in $\mathbb{C}$. The interesting values of r occur when χ_r is at least a binomial and thus also has roots $\neq 0$ in $\mathbb{C}$. These values of r correspond to the slopes of the Newton polygon of L and are also known as *dicritical values* or *weights* [SP03] section 3.3, p. 90. The *dicritical weighted local exponents* of L with respect to r are defined as the non-zero roots of χ_r in $\mathbb{C}$. We set

$$\Omega_r = \{\rho \in \mathbb{C},\, \chi_r(\rho) = 0\},$$
$$\Omega_r^* = \Omega_r \setminus \{0\} = \{\rho \in \mathbb{C}^*,\, \chi_r(\rho) = 0\}.$$

Here, Ω_r^* is non-empty if and only if r is dicritical for L. Merkl then proves

Lemma 2.19. *The number of classical local exponents of L plus the number of dicritical weighted local exponents of L with respect to rational weights $r > 0$, both counted with their multiplicities, equals n, the order of L.*

In the case of a regular singularity, all local exponents are classical and no weighted local exponents appear. So we will assume henceforth that there is at least one weighted local exponent ρ, for some dicritical $r \in \mathbb{Q}^*$. Choose and then fix such a pair.

After these preparations, the first step in the algorithm is to replace in the differential equation $Ly = 0$ the variable y by

$$\exp(-\frac{\rho}{r} x^{-r}) y = e^{-\frac{\rho}{r} x^{-r}} y.$$

This substitution corresponds to a conjugation of L with the multiplication operator given by the indicated exponential function. If we write

$$L = \sum_{j=0}^{n} a_j(x)\delta_r^j$$

the conjugated operator is, see [Mer22] p. 13., given as

$$\widetilde{L} = \sum_{j=0}^{n} \left(\sum_{k=j}^{n} \binom{k}{j} \rho^{k-j} a_j(x) \right) \delta_r^j.$$

It is then shown that the conjugation associated to a weighted local exponent ρ of weight $r > 0$ translates the weighted local exponents of L by ρ, i.e., $\widetilde{L}$ has weighted local exponents $\sigma - \rho$ with respect to r [Mer22], Prop. 3.10, p. 29. In particular, the original ρ becomes 0 and is thus no longer dicritical for $\widetilde{L}$. Iterating this process of conjugation one arrives at a differential equation which has no dicritical weights at all. This is equivalent to saying that the final differential operator L^* has a regular singularity at 0. Thus the normal form theorem 2.15 applies to L^* and produces as many linearly independent solutions of $L^*y = 0$ as its order indicates, using Theorem 2.16. Tracing back the conjugations of L and varying the algorithm over all dicritical weights r and their weighted local exponents ρ, one ends up with a full basis of solutions of the original differential equation. This reproves in a constructive and systematic way Fabry's theorem about the existence and description of the solutions of irregular singular differential equations.

Example 2.20. The divergent series $y(x) = \sum_{k=0}^{\infty} k! x^{k+1}$ satisfies the second order equation

$$Ly = x^3 y'' + (x^2 - x)y' + y = 0.$$

The initial form of L at 0 is given by the first order operator $L_0 = -x\partial + 1$. Hence 0 is an irregular singularity of L. The function $z(x) = \exp(-\frac{1}{x})$ is a second solution of $Ly = 0$; it is no longer a formal power series.

Apparent singularities. The formulas for the solutions of $Ly = 0$ are somewhat complicated whenever the sets Ω of local exponents are not single valued. But if $\Omega = \{\rho\}$ has just one element ρ, i.e., no other local exponent of L is congruent to ρ modulo $\mathbb{Z}$, and if ρ has multiplicity m_ρ, the respective solutions are simpler, of the form, for $0 \leq i < m_\rho$,

$$y_{\rho,i}(x) = x^\rho [f_i + \ldots + f_i \log(x)^i].$$

If some local exponents have multiplicity ≥ 2 logarithms are forced to appear. If all local exponents are simple roots of the indicial polynomial, it may happen that no logarithms appear in the solutions. This situation is known as the presence of *apparent singularities.*

Theorem 2.21 (Apparent singularities). *Let $L \in \mathcal{O}[\partial]$ be a differential operator with holomorphic coefficients and regular singularity at* 0*. Assume that all local exponents are integers and simple roots of the indicial polynomial of L at* 0*, and write $L = L_0 - T$ with initial form L_0 of L. If* $\mathrm{Im}(T) \subseteq \mathrm{Im}(L_0)$ *in $\mathcal{O}$, the local solutions of $Ly = 0$ at* 0 *are holomomorphic functions.*

Proof. This is an immediate consequence of the proof of the normal form theorem, since in case $\mathrm{Im}(T) \subseteq \mathrm{Im}(L_0)$ no extensions of the differential operators to larger function spaces involving logarithms are needed. As the local exponents are integral, the assertion follows from the description of the solutions. □

Gevrey series. By a theorem of Maillet, every power series solution $y(x)$ of an equation $Ly = 0$ with holomorphic coefficients is a *Gevrey-series*, i.e., there exists an $m \in \mathbb{N}$ such that the m-th Borel transform

$$y(x) = \sum_{k=0}^{\infty} a_k x^k \mapsto \widetilde{y}(x) = \sum_{k=0}^{\infty} \frac{a_k}{(k!)^m} x^k$$

of $y(x)$ converges [Mai03]. This result can also be seen as a consequence of the normal form theorem: It suffices to apply the norm estimates in part (ii) of the convergence proof to the series $\widetilde{h}(x) = \sum_{k=0}^{\infty} \frac{a_k}{(k!)^m} x^k$ with $m = n - n'$, where n' denotes again the order of the initial form L_0 of L at 0. Exploiting this one proves that the composition of the automorphism $v = u^{-1}$ of $\widehat{\mathcal{O}}$ with the m-th Borel transform sends the solutions x^ρ of $L_0 y = 0$, for $\rho \in \mathbb{Z}$ a local integer exponent of L, to a convergent power series $x^\rho \widetilde{h}(x)$. The key step is to see that the ratio

$$\frac{(\rho+k)^{\underline{j}}}{\chi_L(\rho+k+i-j)} = \frac{(\rho+k)^{\underline{j}}}{\sum_{\ell=0}^{n} c_{\ell,\ell}(\rho+k+i-j)^{\underline{\ell}}}$$

will be replaced by

$$\frac{(\rho+k)^{\underline{j}}}{\chi_L(\rho+k+i-j)} = \frac{(\rho+k)^{\underline{j}}}{\sum_{\ell=0}^{n'} c_{\ell,\ell}(\rho+k+i-j)^{\underline{\ell}}(k!)^m}$$

to obtain the required convergence. We omit the details. □

3 Differential equations in positive characteristic

3.1 The lack of primitives in characteristic p

From now on let $\Bbbk$ be a field of characteristic $p > 0$. If we try to transfer the description of a basis of solutions of differential equations over $\mathbb{C}$ to fields of characteristic p, substantial obstructions occur, as the following example shows.

Example 3.1. (i) Let $n \in \mathbb{N}$ and let, for $S_{i,j}$ the Stirling numbers,

$$L = \delta^n = (x\partial)^n = x^n\partial^n + S_{n,n-1}x^{n-1}\partial^{n-1} + S_{n,2}x^{n-2}\partial^{n-2} + \ldots + S_{n,1}x\partial + S_{n,0}.$$

If we interpret L as a differential operator in $\mathbb{C}[\![x]\!][\partial]$ and solve the equation $Ly = 0$ in $\mathbb{C}(\!(x)\!)[z]$, we obtain a full basis of solutions $\{1, z, \ldots, z^{n-1}\}$ over $\mathbb{C}$. In characteristic p the field of constants clearly contains $\Bbbk(\!(x^p)\!)[z^p]$. So for $n > p$ the set $\{1, z, \ldots, z^{n-1}\}$ cannot be a full basis of solutions, as 1 and z^p are linearly dependent over the field of constants. In some sense this boils down to the fact that a primitive of z^{p-1} cannot be expressed in terms of z^p, in fact, z^{p-1} does not have a primitive in $\Bbbk(\!(x)\!)[z]$ at all.

(ii) Consider the Euler operator

$$L = x^2\partial^2 + x\partial + 2.$$

Solving $Ly = 0$ in characteristic 0 we notice that the local exponents are given by $\sqrt{2}$ and $-\sqrt{2}$ and a basis of solutions is given by the "functions" $x^{-\sqrt{2}}$ and $x^{\sqrt{2}}$, which are defined in sectors without a branch cut of the logarithm around 0. In $\mathbb{F}_7(\!(x)\!)$ the monomials x^3 and x^4 are solutions of the equation. However, in $\mathbb{F}_5$ no square root of 2 exists and thus it is impossible to solve the Euler equation $Ly = 0$ in $\mathbb{F}_5(\!(x)\!)[z]$.

In order to resolve this issue in positive characteristic, we will construct in the next section a differential extension of $\Bbbk(\!(x)\!)(z)$ which will contain a full basis of solutions for any linear differential operator with regular singularity at 0. Regularity is again needed to ensure the existence of as many local exponents as the order of the differential equation indicates. The extension will overcome the two aforementioned difficulties: the lack of primitives of certain elements and the lack of solutions to Euler equations.

3.2 The Euler-primitive closure $\mathcal{R}$ of $\Bbbk(\!(x)\!)$

For each $\rho \in \Bbbk$ let t^ρ be a symbol. It will play the role of the monomial x^ρ from before; if ρ lies in the prime field of $\Bbbk$ we may substitute x for t to recover the classical setting. We will call ρ the exponent of t in t^ρ. Further, let

$$\mathcal{R} = \bigoplus_{\rho \in \Bbbk} t^\rho \Bbbk(z_1, z_2, \ldots)(\!(x)\!)$$

be the direct sum of Laurent series in x with coefficients in the field of rational functions over $\Bbbk$ in countably many variables z_i, multiplied with the monomials t^ρ. We will simply write $\Bbbk(z)$ and $\Bbbk(z^p)$ instead of $\Bbbk(z_1, z_2, \ldots)$ and $\Bbbk(z_1^p, z_2^p, \ldots)$.

We consider $\mathcal{R}$ as a ring with respect to the obvious addition and the multiplication given by

$$(t^\rho f) \cdot (t^\sigma g) = t^{\rho+\sigma}(f \cdot g)$$

for $\rho, \sigma \in \Bbbk$, $f, g \in \Bbbk[z][\![x]\!]$. In other words, we form the group algebra of the additive group of $\Bbbk$ over $\Bbbk(z)((x))$. We will write $t^0 = 1$ and accordingly have $(t^\rho)^p = t^{\rho p} = 1$ and $\mathcal{R}^p = \Bbbk(z^p)(\!(x^p)\!)$.

Equip $\mathcal{R}$ with the derivation $\partial = \partial_R$ satisfying:

$$\partial x = 1,$$
$$\partial t = t\frac{1}{x},$$
$$\partial t^\rho = \rho t^\rho \frac{1}{x},$$
$$\partial z_1 = \frac{1}{x}, \quad \partial z_2 = \frac{1}{x}\frac{1}{z_1}, \quad \partial z_k = \frac{1}{x}\frac{1}{z_1 \cdots z_{k-1}} = \frac{\partial z_{k-1}}{z_{k-1}},\ k \geq 1.$$

This turns $\mathcal{R}$ into a differential ring.

The action of ∂ on z_i is chosen to mimic the usual derivation of the i-fold composition $\log(\ldots(\log(x))\ldots)$ of the complex logarithm with itself.Indeed we have, writing $\log^{[i]}$ for the i-fold repetition of the logarithm

$$\left(\log^{[i]}(x)\right)' = \frac{1}{x \cdot \log(x) \cdot \log(\log(x)) \cdots \log^{[i-1]}(x)}.$$

Similar constructions with iterated logarithms in positive characteristic were already considered by Dwork [Dwo90], p. 752.

Remark 3.2. (i) The ring $\mathcal{R}$ is not an integral domain. Indeed, $(1+t+\ldots+t^{p-1})(1-t) = 1-t^p = 0$. Thus we are not able to form its quotient field and use the machinery of differential fields, as e.g. Wronski's Lemma and the concept of a basis of solutions. Still, in the course of the next sections, we will be able to provide a precise description of the solutions of a differential equation $Ly = 0$ in $\mathcal{R}$.

(ii) The derivation ∂ leaves the summands of the direct sum $\mathcal{R}$ invariant, i.e., one has $\partial\left(t^\rho \Bbbk(z)(\!(x)\!)\right) \subseteq t^\rho \Bbbk(z)(\!(x)\!)$. This is the reason for not simply defining $\partial t^\rho = \rho t^{\rho-1}$ but rather $\partial t^\rho = \rho t^\rho \frac{1}{x}$.

(iii) Note that the elements of $\mathcal{R}$ may have unbounded degree in each of the variables z_i, only the coefficient of a given power of x has finite degree. This differs from the situation in characteristic 0 where the exponent of the logarithm in a solution of the equation $Ly = 0$ is bounded for each differential operator.

(iv) The doubly iterated logarithm $\log(\log(x))$ of characteristic 0 does not satisfy any homogeneous linear differential equation with holomorphic coefficients, but only the non-linear equation

$$xy'' + y' + x(y')^2 = 0.$$

Alternatively, it satisfies the inhomogeneous equation

$$x\log(x)y' = 1$$

in which the logarithm appears as a coefficient. In characteristic p this reads as

$$xz_1z_2' = 1.$$

For elements of $\mathcal{R}$ the exponents of x are integers, while the exponents of t are elements of the field $\Bbbk$ (formally, t^ρ for $\rho \in \Bbbk$ is just a symbol). However, we will see that the exponents of x and t interact in a certain way. We will use the following convention: In case that ρ is in the prime field $\mathbb{F}_p$ of $\Bbbk$, we may write $x^{\rho_\mathbb{Z}}$ for x^ρ where $\rho_\mathbb{Z} \in \{0, 1, \ldots, p-1\}$ is a representative of ρ. Conversely we may write $t^{k_\Bbbk}$ for t^k for some $k \in \mathbb{Z}$, where $k_\Bbbk \in \mathbb{F}_p \subseteq \Bbbk$ is the reduction of k modulo p.

Before we proceed, we will determine the constants of $\mathcal{R}$. Denote by $\Bbbk(z^p)$ the subfield $\Bbbk(z_1^p, z_2^p, \ldots)$ of $\Bbbk(z)$. A simple computation shows that monomials of the form $t^\rho z_i^p x^{mp-\rho}$, for any ρ in the prime field $\mathbb{F}_p$ of $\Bbbk$ and any $m \in \mathbb{Z}$, are annihilated by ∂. And, actually, these monomials already yield the entire field of constants, namely,

Proposition 3.3. *The ring of constants of $(\mathcal{R}, \partial)$ is*

$$\mathcal{C} := \bigoplus_{\rho \in \mathbb{F}_p} t^\rho x^{p-\rho} \Bbbk(z^p)(\!(x^p)\!),$$

where $\mathbb{F}_p$ denotes the prime field of $\Bbbk$. Moreover, $\mathcal{C}$ is a field.

Proof. Let $f \in \bigoplus_{\rho \in \Bbbk} t^\rho \Bbbk(z)(\!(x)\!)$ and assume that

$$\partial f = 0.$$

Taking derivatives in $\mathcal{R}$ preserves the summands of the direct sum, so it suffices to find constants of the form $t^\rho h$ for some $\rho \in \Bbbk$ and $h \in \Bbbk(z)(\!(x)\!)$.

Fix some $\rho \in \Bbbk$. As for all $k \in \mathbb{Z}$ the derivation ∂ maps $t^\rho \Bbbk(z) x^k$ into $t^\rho \Bbbk(z) x^{k-1}$ by definition, it further suffices to find constants of the form $t^\rho h x^k$, where $h \in \Bbbk(z)$. Therefore we are reduced to search for elements $t^\rho h x^k$ of $\mathcal{R}$ with $\partial(t^\rho h x^k) = 0$. Write $h = g_1/g_2$ for $g_1, g_2 \in \Bbbk[z]$. Then $\partial(t^\rho h x^k) = 0$ is equivalent to $\partial(t^\rho g_1 g_2^{p-1} x^k) = 0$, as g_2^p is a constant. So without loss of generality we may assume that $h \in \Bbbk[z]$ is a polynomial. We expand:

$$0 = \partial(t^\rho h x^k) = t^\rho((\partial h)x + (k+\rho)h)x^{k-1}. \tag{1}$$

Let l be minimal such that $h \in \Bbbk[z_1, \ldots, z_l]$. Consider one monomial $z^\alpha = z_1^{\alpha_1} \cdots z_l^{\alpha_l}$ of h, whose exponent α is maximal among the monomials of h with respect to the component-wise ordering of $\mathbb{N}^l$. Taking the derivative ∂ of a monomial in z decreases the exponents of at least one of the z_i and does not increase the other. It therefore yields a sum of smaller monomials with respect to the chosen ordering.

Thus, in $x\partial h$ the coefficient of z^α vanishes by the maximality of the exponent. If we compare coefficients of $t^\rho x^{k-1} z^\alpha$ in Equation (1) we get $k + \rho = 0$. So it follows that $\rho \in \mathbb{F}_p$ and that $k \equiv \rho \mod p$. Moreover, we see from Equation (1) that $\partial h = 0$. This is clearly equivalent to $h \in \Bbbk[z^p]$. Together with the reductions from above this proves that the ring of constants of $\mathcal{R}$ is indeed

$$\bigoplus_{\rho \in \mathbb{F}_p} t^\rho x^{p-\rho} \Bbbk(z^p)(\!(x^p)\!).$$

Finally, we show that $\mathcal{C}$ is a field. Let

$$f = \sum_{\rho \in \mathbb{F}_p} t^\rho x^{p-\rho} f_\rho \in \mathcal{C},$$

where $f_\rho \in \Bbbk(z^p)(\!(x^p)\!)$. Then we have

$$f^p = \sum_{\rho \in \mathbb{F}_p} t^{p\rho} x^{p^2 - p\rho} f_\rho^p = \sum_{\rho \in \mathbb{F}_p} x^{p^2 - p\rho} f_\rho^p \in \Bbbk(z^p)(\!(x^p)\!),$$

where $f_\rho^p \in \Bbbk(z^{p^2})((x^{p^2}))$. The element f^p vanishes precisely if f_ρ vanishes for all $\rho \in \mathbb{F}_p$, as the exponents of x in each of the summands are from a different residue class modulo p^2. Thus, f^p is a unit for all $f \neq 0$ and we see that $(f^{p-1})(f^p)^{-1}$ is an inverse to f. □

Example 3.1 (cont.) Let us come back to Example 3.1 (i) with $k = p+1$ and the operator $L = (x\partial)^{p+1} \in \Bbbk[x][\partial]$. In $\mathcal{R}$ we have

$$(x\partial)^{p+1}(z_1^p z_2) = 0.$$

So we have found another solution to the equation $Ly = 0$. This completes a basis of a $p+1-$dimensional vector space of solutions over the constants of $\mathcal{R}$, namely $\{1, z_1^1, z_1^2, \ldots, z_1^{p-1}, z_1^p z_2\}$, as those elements are $\mathcal{C}$-linearly independent.

For the Euler operator $L = x^2\partial^2 + x\partial + 2 \in \mathbb{F}_5[x][\partial]$ from Example 3.1 (ii) we can also find a basis of solutions in $\mathcal{R}$ over $\mathcal{C}$. It is given by the monomials t^ω and $t^{-\omega}$, where $\omega \in \mathbb{F}_{25}$ is a square root of 2.

From what we have seen it is reasonable to call $\mathcal{R}$ the *Euler-primitive closure* of $\Bbbk((x))$.

3.3 Extensions of Euler operators to the ring $\mathcal{R}$

Our goal now is to prove that Euler operators admit "enough" solutions in $\mathcal{R} = \bigoplus_{\rho \in \Bbbk} t^\rho \Bbbk(z)((x))$ and then to compute these solutions. For this we first investigate how Euler operators act on monomials $t^\rho z^\alpha x^k$, see Lemma 3.6. For a multi-index $\alpha \in \mathbb{Z}^{(\mathbb{N})} = \{(\alpha_i)_{i\in\mathbb{N}} \mid \alpha_i = 0 \text{ for almost all } i\}$ we write z^α for $z_1^{\alpha_1} \cdots z_n^{\alpha_n}$, if $\alpha_j = 0$ for $j > n$. We define a partial ordering on $\mathbb{Z}^{(\mathbb{N})}$ by $\beta \prec_e \alpha$ if

$$e(\beta) := \overline{\beta}_1 + p\overline{\beta}_2 + p^2\overline{\beta}_3 + \ldots < \overline{\alpha}_1 + p\overline{\alpha}_2 + p^2\overline{\alpha}_3 + \ldots =: e(\alpha),$$

where $\overline{\beta}_i, \overline{\alpha}_i \in \{0, 1, \ldots, p-1\}$ are chosen such that $\beta_i \equiv \overline{\beta}_i \mod p$ respectively $\alpha_i \equiv \overline{\alpha}_i \mod p$. In other words $\prec_e$ is induced by the inverse lexicographic ordering on $\mathbb{F}_p^{(\mathbb{N})}$ via the element-wise reduction modulo p of elements of $\mathbb{Z}^{(\mathbb{N})}$.

We also write $z^\beta \prec_e z^\alpha$ if $\beta \prec_e \alpha$.

Lemma 3.4. *Let $\alpha \in \mathbb{Z}^{(\mathbb{N})}$. Then $(x\partial)z^\alpha$ is a sum of monomials that are smaller than z^α with respect to $\prec_e$ and there is exactly one summand z^γ with $e(\gamma) = e(\alpha) - 1$. In particular, $e(\alpha)$ is the minimal number j such that $(x\partial)^j(z^\alpha) = 0$.*

Proof. Let $\alpha = (\alpha_1, \alpha_2, \ldots)$. We compute:

$$\partial z^\alpha = \frac{1}{x}\sum_{i=1}^{t} \alpha_i \underbrace{z_1^{\alpha_1 - 1} z_2^{\alpha_2 - 1} \cdots z_i^{\alpha_i - 1} z_{i+1}^{\alpha_{i+1}} \cdots z_t^{\alpha_t}}_{=: z^{\gamma(i)}}.$$

If $\alpha_i \not\equiv 0 \mod p$, then clearly $\gamma(i) \prec_e \alpha$, otherwise its coefficient in $(x\partial)z^\alpha$ vanishes. A fast computation shows that if j is the least index, such that $\alpha_j \neq 0$, then $e(\gamma(j)) = e(\alpha) - 1$. Moreover, $e(\gamma(j)) < e(\alpha) - 1$ for all other j. This proves in particular that $e(\alpha)$ is the minimal number j such that $(x\partial)^j z^\alpha = 0$. □

Let s be a variable and $k \in \mathbb{N}$. We define the *j-th Hasse derivative* or *divided derivative* of s^k by $(s^k)^{[j]} = \binom{k}{j} s^{k-j}$; extend it linearly to $\Bbbk[s]$ [Jeo11]. We will apply it below to the indicial polynomial χ_L of an operator L, viewed as a polynomial in the variable s. The next three lemmata are, as in the case of characteristic zero, inspired by Frobenius' "differentiation with respect to local exponents" [Fro73]. See Lemmata 2.6 and 2.7 for the corresponding results in characteristic zero.

Lemma 3.5. *Let $k, l \in \mathbb{N}$. Then we have*

$$(s^{\underline{k}})^{[l]} + (s^{\underline{k}})^{[l+1]}(s-k) = (s^{\underline{k+1}})^{[l+1]}.$$

Lemma 3.6. *Let $j \in \mathbb{N}, k \in \mathbb{Z}, \alpha \in \mathbb{Z}^{(\mathbb{N})}$. Then we have*

$$\partial^j(t^s x^k z^\alpha) = t^s x^{k-j}\left((s+k)^{\underline{j}} z^\alpha + ((s+k)^{\underline{j}})^{[1]} x\partial z^\alpha + \ldots + ((s+k)^{\underline{j}})^{[j]}(x\partial)^j z^\alpha\right).$$

Proof. The proof uses induction on j. For $j = 0$ the claim is obvious. Assume now the formula holds for some $j \geq 0$. Applying ∂ yields

$$\begin{aligned}
\partial^{j+1}(t^s x^k z^\alpha) &= \partial\left(t^s x^{k-j}\left((s+k)^{\underline{j}} z^\alpha + ((s+k)^{\underline{j}})^{[1]} x^1 \partial z^\alpha + \ldots + ((s+i)^{\underline{j}})^{[j]}(x\partial)^j z^\alpha\right)\right) \\
&= t^s x^{k-j-1}(s+k-j)\left((s+k)^{\underline{j}} z^\alpha + ((s+k)^{\underline{j}})^{[1]} x\partial z^\alpha + \ldots + ((s+k)^{\underline{j}})^{[j]}(x\partial)^j z^\alpha\right) + \\
&\quad + t^s x^{k-j}\left((s+k)^{\underline{j}} \partial z^\alpha + ((s+k)^{\underline{j}})^{[1]} \partial(x\partial) z^\alpha + \ldots + ((s+k)^{\underline{j}})^{[j]}\partial(x\partial)^j z^\alpha\right) \\
&= t^s x^{k-j-1}\left((s+k-j)(s+k)^{\underline{j}} + \left((s+k-j)((s+k)^{\underline{j}})^{[1]} + (s+k)^{\underline{j}}\right) x\partial z^\alpha + \ldots\right) \\
&= t^s x^{k-j-1}\left((s+k)^{\underline{j+1}} z^\alpha + ((s+k)^{\underline{j+1}})^{[1]} x\partial z^\alpha + \ldots + ((s+k)^{\underline{j+1}})^{[j+1]}(x\partial)^{j+1} z^\alpha\right),
\end{aligned}$$

where we have used the previous lemma in the last step. ☐

From this we get:

Lemma 3.7. *Let L be an Euler operator of order n with indicial polynomial χ_L. Then for any $\alpha \in \mathbb{Z}^{(\mathbb{N})}$, $k \in \mathbb{Z}$ and $\rho \in \Bbbk$ we have*

$$L(t^\rho x^k z^\alpha) = t^\rho x^k\left(\chi_L(\rho+k) z^\alpha + \chi_L'(\rho+k) x\partial(z^\alpha) + \ldots + \chi_L^{[n]}(\rho+k)(x\partial)^n(z^\alpha)\right). \tag{2}$$

For a field K of characteristic 0 a polynomial $q \in K[s]$ has a j-fold root at $\beta \in \overline{K}$ if and only if the first $j-1$ derivatives of q vanish in β, but the j-th derivative does not. This very statement is false in characteristic p, but if one replaces derivatives with Hasse derivatives it holds true.

Lemma 3.8. *Let $q \in \Bbbk[s]$ be a polynomial. Then a is an j-fold root of q if and only if $q^{[i]}(a) = 0$ for $i < j$, but $q^{[j]}(a) \neq 0$.*

With these results we can finally solve Euler equations in the ring $\mathcal{R}$. We prove that, similar to the complex case in Lemma 2.8, the solutions form a vector space of dimension n over the constants $\mathcal{C} \subseteq \mathcal{R}$.

Proposition 3.9. *Let L be an Euler operator of order n and let $\Omega := \{\rho_1, \ldots, \rho_k\}$ be the set of local exponents of L at 0 with multiplicities $m_{\rho_1}, \ldots, m_{\rho_k}$. The solutions of $Ly = 0$ in $\mathcal{R}$ form a $\mathcal{C}$-subspace of dimension n. A basis is given by*

$$y_{\rho,i} := t^\rho z^{i^*}, \quad \rho \in \Omega,\ i < m_\rho,$$

where

$$i^* = (i, \lfloor i/p \rfloor, \lfloor i/p^2 \rfloor, \lfloor i/p^3 \rfloor, \ldots) \in \mathbb{Z}^{(\mathbb{N})}.$$

Before we prove the proposition let us consider an example.

Example 3.10. Consider the differential operator $L = x^6\partial^6 + x^4\partial^4 + x^3\partial^3 + x^2\partial^2 \in \mathbb{F}_2[x][\partial]$ with indicial polynomial $\chi_L(s) = (s-1)^5 s$. As the operator has order 6 one expects 6 solutions of $Ly = 0$, independent over $\mathcal{C}$. The proposition asserts that a basis is given by

$$1, x, xz_1, xz_1^2 z_2, xz_1^3 z_2, xz_1^4 z_2^2 z_3.$$

Indeed, one easily verifies that all these monomials are solutions and are $\mathcal{C}$-linearly independent.

Proof. The operator L is $\mathcal{C}$-linear and maps $t^\rho x^k \Bbbk(z)$ into itself. Therefore it suffices to find solutions of $Ly = 0$ of the form $t^\rho f(z) x^k$, where $f \in \Bbbk(z)$. Further we can argue similar as in Proposition 3.3: we write $f = g_1/g_2$ for $g_1, g_2 \in \Bbbk[z]$. If $t^\rho f(z) x^k$ is a solution, then so is

$$g_2(z)^p \left(t^\rho f(z) x^k\right) = t^\rho g_1(z) g_2(z)^{p-1} x^k,$$

as $g_2^p \in \Bbbk[z^p] \subseteq \mathcal{C}$. So we may assume without loss of generality that $0 \neq f \in \Bbbk[z]$.

Let z^α be the largest monomial of $f(z)$ with respect to the ordering $\prec_e$. By Lemma 3.7 and the linearity of L we obtain

$$L(t^\rho f(z) x^k) = t^\rho \left(\chi_L(\rho+k) f(z) + \chi_L^{[1]}(\rho+k)(x\partial) f(z) + \ldots + \chi_L^{[n]}(\rho+k)(x\partial)^n f(z)\right).$$

Hence $L(t^\rho f(z) x^k)$ vanishes if and only if

$$\chi_L(\rho+k) f(z) + \chi_L^{[1]}(\rho+k)(x\partial) f(z) + \ldots + \chi_L^{[n]}(\rho+k)(x\partial)^n f(z)$$

vanishes. We compare the coefficients of monomials in z starting with the largest. All appearing monomials are smaller than or equal to z^α by Lemma 3.4 and for all monomials z^γ in the summand $\chi_L(\rho+k)(x\partial)^j$ we have $e(\gamma) \leq e(\alpha) - j$. So in order for the sum to vanish, $\chi_L(\rho+k)$ has to vanish by comparing coefficients of z^α. Further, by comparing coefficients of the next smaller monomials, we obtain $\chi_L^{[1]}(\rho+k) = 0$ or $(x\partial) z^\alpha = 0$, i.e. $e(\alpha) = 1$. Inductively we obtain that the sum vanishes, if and only if $\chi_L^{[\ell]}(\rho+k) \neq 0$ implies that $e(\alpha) < \ell$. Put differently, by Lemma 3.8, if $\rho+k$ is a local exponent of L of multiplicity $m_{\rho+k}$, then $e(\alpha) < m_{\rho+k}$. Thus we can give a complete description of the elements in the kernel of L. They are of the form $t^\rho x^k z^\alpha$, where $e(\alpha) < m_{\rho+k}$.

A quick calculation using Lemma 3.4 shows that the last condition is fulfilled for multi-indices, whose entries differ by multiples of p from i^* for $i = 0, \ldots, m_{\rho+k} - 1$. This shows on the one hand that the elements $y_{\rho,i}$ are indeed solutions of $Ly = 0$. On the other hand, the elements $y_{\rho,i}$ are chosen such that ρ ranges over all local exponent of L exactly once. For $\rho + k$ to be a local exponent, i.e., a zero of χ_L, we may add multiples of p to k, or subtract an element of the prime field from ρ and add it to k. Those transformations can be realized by multiplying a solution $t^\rho x^k f(z)$ by an element from $\mathcal{C}$. So indeed, all solutions of $Ly = 0$ are linear combinations of the elements $y_{\rho,i}$; that is they generate the $\mathcal{C}$-vector space of solutions.

Assume now that a $\mathcal{C}$-linear relation between the solutions $y_{\rho,i}$ exists. Let

$$\mathcal{D} := \bigoplus_{\rho \in \mathbb{F}_p} t^\rho x^{p-\rho} \Bbbk[z^p][\![x^p]\!].$$

As $\mathcal{C} = \mathrm{Quot}\mathcal{D}$, it suffices to consider a relation with coefficients in $\mathcal{D}$. Let $\Omega = \bigsqcup_j \Omega_j$ be the set of all local exponents, where two local exponents ρ, σ are in the same subset Ω_j if and only if their difference is in the prime field. Assume that

$$\sum_j \sum_{\substack{\rho \in \Omega_j \\ i < m_\rho}} y_{\rho,i} \cdot d_{\rho,i} = 0$$

for some $d_{\rho,i} \in \mathcal{D}$. As the exponents of t of elements of $\mathcal{D}$ are in the prime field of $\Bbbk$, it follows that for each j the sum

$$\sum_{\substack{\rho \in \Omega_j \\ i < m_\rho}} y_{\rho,i} \cdot d_{\rho,i}$$

vanishes. So it suffices to focus on relations between solutions corresponding to local exponents in the same set set Ω_j. Without loss of generality $\Omega_j = \mathbb{F}_p$, the prime field of $\Bbbk$. We consider now a relation of the form

$$\sum_{\substack{\rho \in \mathbb{F}_p \\ i < m_\rho}} y_{\rho,i} \cdot d_{\rho,i} = 0.$$

Without loss of generality we may assume that at least one of the constants $d_{\rho,i}$ has order 0 in x and let $f_{\rho,i} \in \Bbbk[z^p]$ be its constant term. Taking the coefficient of the monomial with smallest degree with respect to x in the sum above, we obtain a relation of the form

$$\sum_{\substack{\rho\in\mathbb{F}_p\\ i<m_\rho}} t^\rho z^{\alpha_i} \cdot f_{\rho,i} = 0.$$

This sum vanishes if and only if the summand for each $\rho \in \mathbb{F}_p$ vanishes. Furthermore the multi-exponents $i^* = (i, \lfloor i/p \rfloor, \lfloor i/p^2 \rfloor, \ldots)$ are defined such that no two of them differ by multiples of p in every component. Thus $f_{\rho,i} = 0$ for all ρ and i, as required.

Finally, note that $\sum_{\rho\in\Omega} m_\rho = n$, as χ_L is a polynomial of degree n. So the dimension of the space of solutions is indeed n. □

3.4 The normal form theorem in positive characteristic

We have seen that a basis of solutions of Euler equations is of a very special form. It is not to be expected that solutions of general differential equations with regular singularities are similarly simple. In the following let ρ be a fixed local exponent of L at 0 of multiplicity m_ρ. We define a function $n_\rho : \mathbb{N} \to \mathbb{N}$

$$n_\rho(0) = m_\rho, \qquad n_\rho(k+1) = n_\rho(k) + m_{\rho+k+1},$$

where $m_{\rho+j} = 0$ if $\rho + j$ is not a root of the indicial polynomial. In other words

$$n_\rho(k) = m_\rho + m_{\rho+1} + \ldots + m_{\rho+k}.$$

Note here that if $k > p$ the summand m_ρ appears at least twice in the sum. Moreover we define the *ρ-function space $\mathcal{F} = \mathcal{F}_L^\rho$ associated to L* as

$$\mathcal{F}_L^\rho := t^\rho \sum_{k=0}^{\infty} \bigoplus_{\alpha\in\mathcal{A}_k} \Bbbk z^\alpha x^k,$$

where

$$\mathcal{A}_k := \left\{ \alpha \in \mathbb{N}^{(\mathbb{N})} \middle| \alpha_1 < n_\rho(k), \alpha_{j+1} \le \alpha_j/p \quad \text{for all } j \in \mathbb{N} \right\}$$

is a finite subset of $\mathbb{N}^{(\mathbb{N})}$. Note that $\mathcal{F}$ only depends on the initial form of the differential operator L, more precisely, only on the multiplicity of all local exponents of L that differ from ρ by an element of the prime field of $\Bbbk$.

Example 3.11. Consider the differential operator $L = x^3\partial^3 + 2x^2\partial^2 + \widetilde{L} \in \mathbb{F}_3[\![x]\!][\partial]$, where $\widetilde{L} \in \mathbb{F}_3[\![x]\!][\partial]$ has positive shift. The local exponents of L are 0 with multiplicity 2 and 1 with multiplicity 1. The monomials in $\mathcal{F}_L^0$ are depicted below in Figure 2.

Lemma 3.12. *Let $L \in \Bbbk[\![x]\!][\partial]$ be a linear differential operator and let ρ be one of its local exponents. The space $\mathcal{F}_L^\rho = \mathcal{F}$ is invariant under all differential operators with non-negative shift. In particular we have $L\mathcal{F} \subseteq \mathcal{F}$.*

Proof. We can rewrite any differential operator with non-negative shift in terms of the operator $\delta = x\partial$ instead of ∂, where the base change between $x^n\partial^n$ and δ^n is given by the Stirling numbers, see Remark 2.1. So we investigate the action of δ on a monomial $t^\rho x^i z^\alpha \in \mathcal{F}$, where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^{(\mathbb{N})}$. We compute as in Lemma 3.4

$$\delta(t^\rho x^k z^\alpha) = x\partial(t^\rho x^k z^\alpha) = t^\rho x^k \left((k+\rho)z^\alpha + \sum_{j=1}^{n} \alpha_j z_1^{\alpha_1-1} \cdots z_j^{\alpha_j-1} z_{j+1}^{\alpha_{j+1}} \cdots z_n^{\alpha_n} \right).$$

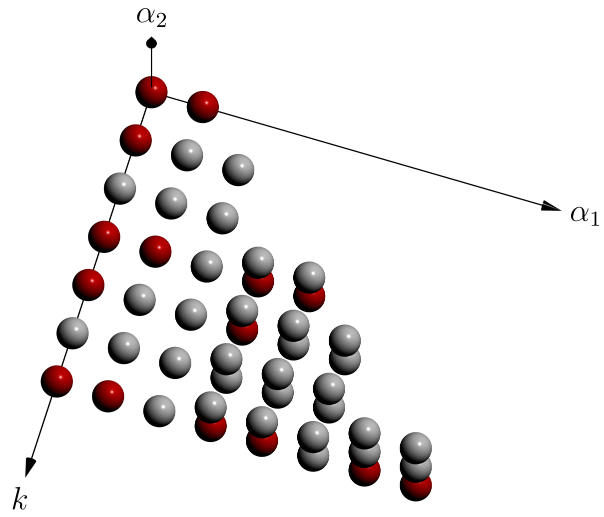

FIGURE 2: The set of exponents (k, α_1, α_2) of monomials $x^k z_1^{\alpha_1} z_2^{\alpha_2}$ in $\mathcal{F}_L^0$ with $k \leq 6$, exponents of monomials in $\mathrm{Ker}(L_0)$ in red. They are $1, z_1, x, x^3, x^3 z_1, x^3 z_1^3, x^3 z_1^4, x^4, x^4 z_1^3, x^6, x^6 z_1, x^6 z_1^3, x^6 z_1^4, x^6 z_1^6, x^6 z_1^7$.

We want to show that all exponents of monomials with non-zero coefficient in the sum above are in $\mathcal{A}_k$. It is clear that $\alpha \in \mathcal{A}_k$ by assumption, so it remains to prove that if $\alpha_j \not\equiv 0 \mod p$ then

$$(\alpha_1 - 1, \ldots, \alpha_j - 1, \alpha_{j+1}, \ldots, \alpha_n) \in \mathcal{A}_k$$

for $j = 1, \ldots, n$. If $\alpha_{l+1} \leq \alpha_l / p$ then also $\alpha_{l+1} - 1 \leq (\alpha_l - 1)/p$ for $l < j$. It remains to show that $\alpha_{j+1} > (\alpha_j - 1)/p$ implies $\alpha_j \equiv 0 \mod p$. For this we see that from

$$\alpha_j - 1 < p\alpha_{j+1} \leq \alpha_j$$

it follows indeed that p divides $\alpha_j = p\alpha_{j+1}$. □

Lemma 3.13 (cf. Lemma 2.13)**.** *Let $L_0 \in \Bbbk[\![x]\!][\partial]$ be an Euler operator with local exponent ρ and associated ρ-function space $\mathcal{F}$. Then $L_0(\mathcal{F}) = \mathcal{F}x$.*

Proof. First we show that any monomial in $\mathcal{F}$ gets mapped to $\mathcal{F}x$ under L_0. Let $t^\rho x^k z^\alpha \in \mathcal{F}$. By Lemma 3.7 we have

$$L_0(t^\rho x^k z^\alpha) = t^\rho x^k \left(\chi_L(\rho + k) z^\alpha + \chi_L'(\rho + k) x\partial(z^\alpha) + \ldots + \chi_L^{[n]}(\rho + k)(x\partial)^n(z^\alpha) \right).$$

By Lemma 3.12 this expression is contained in $\mathcal{F}$. The first $m_{\rho+k}$ summands of the sum vanish due to Lemma 3.8. In the remaining summands $x\partial$ is applied at least $m_{\rho+k}$ times to z^α, decreasing the exponent of z_1 by at least $m_{\rho+k}$. Thus for each monomial with non-zero coefficient in

$$\chi_L^{[m_{\rho+k}]}(\rho + k)(x\partial)^{m_{\rho+k}}(z^\alpha) + \ldots + \chi_L^{[n]}(\rho + k)(x\partial)^n(z^\alpha)$$

the exponents of z are in $\mathcal{A}_{k-1}$ and thus $L_0(t^\rho x^k z^\alpha) \in \mathcal{F}x$.

Now we show that every monomial of $\mathcal{F}x$ is in the image of $\mathcal{F}$ under L_0. We proceed by induction on $e(\alpha) = \overline{\alpha_1} + p\overline{\alpha_2} + p^2\overline{\alpha_3} + \ldots$ Let $t^\rho x^{k+1} z^\alpha \in \mathcal{F}x$; that is $\alpha \in \mathcal{A}_k$. Assume that $\rho + k + 1$ is an ℓ-fold root of χ_L, where ℓ is set equal to 0 if $\rho + k + 1$ is not a root at all. We define an element $\beta \in \mathcal{A}_{k+1}$ such that $L_0(t^\rho x^{k+1} z^\beta) = t^\rho x^{k+1} z^\alpha + r$, where r is a sum of smaller monomials with respect to $\prec_e$. Set

$$\beta_1 = \alpha_1 + \ell, \qquad \beta_j = \alpha_j + \lfloor \beta_{j-1}/p \rfloor - \lfloor \alpha_{j-1}/p \rfloor.$$

As $t^\rho x^k z^\alpha \in \mathcal{F}$, we have $\alpha_1 < n_\rho(k)$ and therefore $\beta_1 = \alpha_1 + \ell < n_\rho(k+1)$. Moreover, we know that $\alpha_j \leq \lfloor \alpha_{j-1}/p \rfloor$ and therefore also $\beta_j = \alpha_j + \lfloor \beta_{j-1}/p \rfloor - \lfloor \alpha_{j-1}/p \rfloor \leq \beta_{j-1}/p$. By construction we have $\beta_1 = \alpha_1 + l$ and thus $\beta_1 < n_\rho(k+1)$. Altogether this proves $\beta \in \mathcal{A}_{k+1}$.

Finally we show that $L_0(t^\rho x^{k+1} z^\beta)$ is of the desired form. Again by Lemma 3.7 we have

$$L_0(t^\rho x^{k+1} z^\beta) = t^\rho x^{k+1} \left(\chi_L(\rho + k + 1) z^\beta + \ldots + \chi_L^{[n]}(\rho + k + 1)(x\partial)^n (z^\beta) \right)$$

As $\rho + k + 1$ has multiplicity ℓ as a zero of χ_L, the first ℓ summands of this expansion vanish, according to Lemma 3.8. If one expands the further summands using the Leibniz rule one gets a sum of monomials of the form $c_\gamma t^\rho x^{k+1} z^\gamma$, with $c_\gamma \in \Bbbk$. The exponents γ are in $\mathcal{A}_k$ and by Lemma 3.4 we have $e(\gamma) \leq e(\beta) - \ell = e(\alpha)$. Only one of these summands fulfils $e(\gamma) = e(\alpha)$. It is of the form $c_\alpha t^\rho x^{k+1} z^\alpha$ by construction. Now by the induction hypothesis, all other summands are in the image of $\mathcal{F}$ under L_0; they are in $\mathcal{F}x$ because of Lemma 3.12. Thus, $t^\rho x^{k+1} z^\alpha \in L_0(\mathcal{F})$, which concludes the proof. □

Remark 3.14. The proof of the surjectivity of L_0 is constructive: For each monomial $t^\rho x^k z^\alpha$ in $\mathcal{F}x$ one constructs a monomial $t^\rho x^k z^\beta$ in $\mathcal{F}$, such that $L_0(t^\rho x^k z^\beta) = c t^\rho x^k z^\alpha + r$, where r is a sum of smaller monomials. If $r = 0$ we divide by c and are done. Otherwise we iterate the construction for all monomials in r and subtract the monomials obtained this way from $c^{-1} t^\rho x^k z^\beta$. After at most $e(\alpha)$ steps $r = 0$ and we have constructed an element of $\mathcal{F}$ which is sent to $t^\rho x^k z^\alpha$ by L_0.

Lemma 3.15. *The kernel of the restriction of L_0 to $\mathcal{F} = \mathcal{F}_L^\rho$ is topologically spanned over $\Bbbk$ by monomials of the form $t^\rho x^k z^\alpha$, where*

$$\alpha \in \mathcal{A}_k = \left\{ \alpha \in \mathbb{N}^{(\mathbb{N})} \middle| \alpha_1 < n_\rho(k), \alpha_{j+1} \leq \alpha_j / p \quad \textit{for all } j \in \mathbb{N} \right\}$$

with $e(\alpha) < m_{\rho+k}$. Consequently, a direct complement $\mathcal{H}$ of $\operatorname{Ker} L_0|_\mathcal{F}$ is topologically spanned by monomials of the form $t^\rho x^k z^\alpha$, where $\alpha \in \mathcal{A}_k$ with $e(\alpha) \geq m_{\rho+k}$.

Proof. We have seen that $e(\alpha)$ is the least number k, such that $(x\partial)^k z^\alpha = 0$. So every monomial $t^\rho x^k z^\alpha$ with $e(\alpha) < m_{\rho+k}$ is in the kernel of L_0 according to Lemma 3.7. Arguing as in the proof of Proposition 3.9 we see that those elements indeed span $\ker L_0|_\mathcal{F}$. □

Now we are ready to state and prove the normal form theorem.

Theorem 3.16 (Normal form theorem in positive characteristic). *Let $\Bbbk$ be an algebraically closed field of characteristic p. Let $L \in \Bbbk[\![x]\!][\partial]$ be a differential operator with initial form L_0 and shift $\tau = 0$ acting on $\mathcal{R} = \bigoplus_{\rho \in \Bbbk} t^\rho \Bbbk(z)(\!(x)\!)$. Let ρ be a local exponent of L at 0 and $\mathcal{F} = t^\rho \sum_{k=0}^\infty \bigoplus_{\alpha \in \mathcal{A}_k} \Bbbk z^\alpha x^k \subset \mathcal{R}$ the associated ρ-function space.*

(i) The map $L_0|_\mathcal{H} : \mathcal{H} \to \mathcal{F}x$ is bijective and the composition of its inverse $(L_0|_\mathcal{H})^{-1} : \mathcal{F}x \to \mathcal{H}$ composed with the inclusion $\mathcal{H} \subseteq \mathcal{F}$ defines a $\mathcal{C}$-linear right inverse $S : \mathcal{F}x \to \mathcal{F}$ of L_0.

(ii) Let $T = L_0 - L : \mathcal{F} \to \mathcal{F}x$. Then the map

$$u = \mathrm{Id}_\mathcal{F} - S \circ T : \mathcal{F} \to \mathcal{F}$$

is a continuous $\mathcal{C}$-linear automorphism of $\mathcal{F}$ with inverse $v = \sum_{k=0}^\infty (S \circ T)^k : \mathcal{F} \to \mathcal{F}$.

(iii) The automorphism v of $\mathcal{F}$ transforms L into L_0, i.e.,

$$L \circ v = L_0.$$

Proof. For (i) note that by Proposition 3.13 the map $L_0 : \mathcal{F} \to \mathcal{F}x$ is surjective and thus the restriction to a direct complement of its kernel is bijective. Clearly S then defines a right inverse of L_0. One easily checks that the construction of preimages of L_0 mentioned in Remark 3.14 is $\mathcal{C}$-linear.

The assertions (ii) and (iii) are an application of the Perturbation Lemma 2.14. We view elements of $\mathcal{F}$ as power series in x and equip $\mathcal{F}$ with the x-adic topology, which turns it into a complete metric space. The operator T has positive shift by definition and thus increases the order in x of a monomial $t^\rho x^k z^\alpha$ and thus of any element of $\mathcal{F}$. The operator S maintains the order in x of a monomial as L_0 does so. So the composition $S \circ T$ increases the order of any element. Furthermore, T maps $\mathcal{F}$ to $\mathcal{F}x = \mathrm{Im}(L_0)$. So we may apply the perturbation lemma and the claim follows. $\square$

3.5 Solutions of regular singular equations

The normal form theorem allows us to describe all solutions of differential equations with regular singularities.

Theorem 3.17. *Let $L \in \Bbbk[\![x]\!][\partial]$ be a linear differential operator with regular singularity at 0 acting on $\mathcal{R}$. Let $\rho \in \Bbbk$ be a local exponent of L. Denote by $u_\rho : \mathcal{F}_L^\rho \to \mathcal{F}_L^\rho$ the automorphism associated to ρ given in (ii) of the normal form theorem. The solutions of the differential equation $Ly = 0$ in $\mathcal{R}$ form an n-dimensional $\mathcal{C}$-vector space. A basis is given by*

$$y_{\rho,i} = u_\rho^{-1}(t^\rho z^{i^*}),$$

where ρ varies over the local exponents of L at 0 and $0 \le i < m_\rho$, with $i^* = (i, \lfloor i/p \rfloor, \lfloor i/p^2 \rfloor, \ldots)$.

Proof. By the normal form theorem and the description of the solutions of Euler equations (Proposition 3.9), we have

$$L(y_{\rho,i}) = L \circ u_\rho^{-1}(t^\rho z^{i^*}) = L_0(t^\rho z^{i^*}) = 0,$$

so these functions clearly are solutions of the differential equation $Ly = 0$. Let now y be any solution of $Ly = 0$. Again, as L commutes with the direct sum decomposition of

$$\mathcal{R} = \bigoplus_{\rho \in \Bbbk} t^\rho \Bbbk(z)(\!(x)\!)$$

and upon multiplication with constants of the form x^{kp} we may assume that y is of the form $y = t^\rho \left(\sum_{k=0}^\infty f_k(z) x^k\right)$ for $f_k \in \Bbbk(z)$. If we write $L = L_0 - T$ we obtain

$$L_0 y - Ty = 0,$$

where T has positive shift, i.e., it strictly increases the order in x. Thus, $t^\rho f_0(z)$ is a solution to the Euler equation $Ly = 0$ and therefore

$$t^\rho f_0(z) = \sum_{(\sigma,i)} c_{\sigma,i} t^\sigma z^{\alpha_i},$$

where σ varies over the local exponents, $0 \le i < m_\sigma$, and $c_{\sigma,i} \in \mathcal{C}$ is homogeneous of order 0 in x. We compute

$$L\left(y - \sum_{(\sigma,i)} c_{\sigma,i} y_{\rho,i}\right) = L\left(-\sum_{(\sigma,i)} c_{\sigma,i} u_\sigma^{-1}(t^\sigma z^{\alpha_i})\right) = -\sum_{(\sigma,i)} c_{\sigma,i} L(u_\sigma^{-1}(t^\rho z^{i^*})) = 0.$$

Note that for all $f \in \mathcal{F}$ we have $\mathrm{ord}_x(f - u(f)) > \mathrm{ord}_x f$, i.e., the monomial of order 0 remains unchanged under u. So $y - \sum_{(\sigma,i)} c_{\sigma,i} y_{\sigma,i}$ has positive order in x. Iteration yields constants $d_{\sigma,i} \in \mathcal{C}$ with $y = \sum_{(\sigma,i)} d_{\sigma,i} y_{\sigma,i}$. Thus, y is a linear combination of $y_{\sigma,i}$. Conversely, any such linear combination is a solution of $Ly = 0$.

The linear independence of the solutions $y_{\rho,i}$ can be reduced to the linear independence of the solutions of the Euler equation, which was proven in Proposition 3.9.

This proves that the solutions of $Ly = 0$ in $\mathcal{R}$ form an n-dimensional $\mathcal{C}$-vector space with basis $y_{\rho,i}$, where ρ varies over the local exponents and $0 \leq i < m_\rho$. □

Remark 3.18. We have assumed for convenience that our field $\Bbbk$ is algebraically closed. If this is not the case, e.g., in the case of a finite field $\mathbb{F}_p$, there is no need to pass to the entire algebraic closure. In the constructions involved in the normal form theorem for an operator L one has to find the roots of the characteristic polynomial $\chi_L \in \Bbbk[s]$, the local exponents ρ. Further we have to evaluate the characteristic polynomial at the values $\rho + k$ for elements k of the prime field of $\Bbbk$. Thus, if χ_L splits over $\Bbbk$ the normal form theorem works without problems within $\Bbbk$. Otherwise it is sufficient to pass to a splitting field of χ_L to describe a full basis of solutions.

Remark 3.19. (i) The space $\mathcal{R}$ provides us with n linearly independent solutions for any operator with a regular singularity at 0 in characteristic p. It is minimal in the following sense: we only introduce a new variable z_i whenever the algorithm constructing solutions forces us to do so, i.e., when we have to divide by p. It is possible to choose a system of representatives $\Lambda \subseteq \Bbbk$ of the set $\Bbbk/\mathbb{F}_p$ of residue classes and to then define

$$\widetilde{\mathcal{R}} := \bigoplus_{\rho \in \Lambda} t^\rho \Bbbk(z)(\!(x)\!).$$

It suffices to construct solutions in $\widetilde{\mathcal{R}}$ of any linear differential equation $Ly = 0$ having a regular singularity at 0, similarly as above. For example, if $\sigma \in \Bbbk$ is a local exponent of an Euler operator and there is $\rho \in \Lambda$ with $\rho + k = \sigma$ for some $\sigma \in \mathbb{F}_p$ and $k \in \mathbb{F}_p$, then $t^\rho x^k$ is a solution of the equation $Ly = 0$. This construction has the advantage that the constants are much simpler, as they are given by the elements of

$$\mathcal{C}_{\widetilde{\mathcal{R}}} = \Bbbk(z^p)(\!(x^p)\!).$$

However, this procedure requires a choice of a system of representatives of $\Bbbk/\mathbb{F}_p$.

(ii) In characteristic 0 a minimal extension of $\Bbbk(\!(x)\!)$ in which every regular singular equation has a full basis of solutions is the universal Picard-Vessiot ring or field for differential equations with regular singularities, discussed in [SP03].

4 Examples and applications in characteristic p

4.1 Examples

Example 4.1 (Exponential function in characteristic 3)**.** We consider the equation $y' = y$. Solving over the holomorphic functions, or in $\mathbb{C}[\![x]\!]$ one obtains the exponential function as a solution. However there is no reduction of this function modulo any prime, as all prime numbers appear in the denominators of the expansion of the exponential function. But one can obtain solutions modulo p for any prime in $\mathcal{R}$ using the normal form theorem. Pick for example $p = 3$. Write $L = x\partial - x = \delta - x$, so our equation is equivalent to $Ly = 0$. The only local exponent of the equation is 0, thus one needs to compute the series

$$\sum_{n=0}^{\infty} (S \circ T)^n(1).$$

The operator T is simply given by the multiplication by x, where S is, as constructed above, a right-inverse of $L_0 = x\partial$. One obtains:

$$\begin{aligned}
(S\circ T)^1(1) &= & S(x) &= & x,\\
(S\circ T)^2(1) &= & S(x^2) &= & 2x^2,\\
(S\circ T)^3(1) &= & S(2x^3) &= & 2x^3z_1,\\
(S\circ T)^4(1) &= & S(2x^4z_1) &= & 2x^4z_1 + x^4,\\
(S\circ T)^5(1) &= & S(2x^5z_1 + x^5) &= & x^5z_1,\\
(S\circ T)^6(1) &= & S(x^6z_1) &= & 2x^6z_1^2,\\
(S\circ T)^7(1) &= & S(2x^7z_1^2) &= & 2x^7z_1^2 + 2x^7z_1 + x^7,\\
(S\circ T)^8(1) &= & S(2x^8z_1^2 + 2x^8z_1 + x^8) &= & x^8z_1^2 + 2x^8,\\
(S\circ T)^9(1) &= & S(x^9z_1^2 + 2x^9) &= & x^9z_1^3z_2 + 2x^9z_1.
\end{aligned}$$

One gets the solution

$$1+x+2x^2+2x^3z_1+x^4(1+2z_1)+x^5z_1+2x^6z_1^2+x^7(1+2z_1+2z_1^2)+x^8(2+z_1^2)+x^9(2z_1+z_1^3z_2)+\ldots,$$

which could be considered as the exponential function in characteristic 3. Note that obtaining the rightmost column needs some computational effort. One has to follow the steps described in Remark 3.14. There seems to be no obvious pattern in the coefficients of the obtained power series.

Similarly, one can compute the exponential functions $\exp_p$ for other characteristics p. For $p = 2$ the first terms are

$$1+x+x^2z_1+x^3(z_1+1)+x^4(z_1^2z_2+z_1)+x^5z_1^2z_2+x^6(z_1^3z_2+z_1^3)+x^7(z_1^3z_2+z_1^2z_2+z_1^3+z_1+1)+\ldots$$

and for $p = 5$ we get

$$1 + x + 3x^2 + x^3 + 4x^4 + 4x^5z_1 + x^6(4z_1 + 1) + x^7(2z_1 + 2) + x^8(4z_1 + 1) + x^9z_1 + 3x^{10}z_1^2 + \ldots$$

The series $\exp_p$ seems to have some remarkable properties. Let us considers the constant term in z, i.e. $\exp_p(x, 0, 0, \ldots)$. Computations suggest that $y = \exp_3(x, 0, 0, \ldots)$ satisfies

$$x^3y^3 + xy^2 - y + 1 = 0$$

and $y = \exp_5(x, 0, 0, \ldots)$ satisfies

$$x^{10}y^5 + x^6y^4 + x^4y^3 - x^3y^3 + 2x^2y^2 + 2xy^2 - 2y + 2 = 0,$$

i.e. these series seem to be algebraic over $\mathbb{F}_p(x)$. Similar observations were made for other characteristics as well. This motivates the following challenge:

Problem 4.2. *Let $L \in \mathbb{F}_p[x][\partial]$ be a differential operator and $\rho \in \mathbb{F}_p$ a local exponent of L. Let u be the automorphism described in the normal form theorem in positive characteristic, Theorem 3.16. Determine the cases where $(u^{-1}(x^\rho))_{|z_i=0}$ is algebraic over $\mathbb{F}_p(x)$.*

In the next example, the answer is immediate.

Example 4.3. We consider the minimal complex differential equation $Ly = 0$ for

$$y(x) = -\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots \in \mathbb{C}[\![x]\!].$$

It is given by $L = x^2\partial^2 - (x^2\partial + x^3\partial^3)$. The local exponents are $0, 1$ and a basis of solutions in $\mathbb{C}[\![x]\!]$ is given by $\{1, y\}$. Reducing L modulo a prime number p one again finds the local exponents $0, 1$. Clearly $y_{0,0} = u_0^{-1}(1) = 1$. Further we compute

$$y_{1,0} = u_1^{-1}(t^1) = \sum_{k=0}^{\infty}(S\circ T)(t^1) = t\left(1 + \frac{x}{2} + \frac{x^2}{3} + \ldots + \frac{x^{p-2}}{p-1} + x^{p-1}z_1\right).$$

Here only adjoining the variable z_1 instead of countably many z_i is necessary to obtain enough solutions. In the next section we will describe the class of operators, where the addition of finitely many of the variables z_i suffice.

4.2 Special cases

Equations with local exponents in the prime field. The situation becomes much easier if we consider a linear differential equation $Ly = 0$, whose local exponents are all contained in the prime field $\mathbb{F}_p \subseteq \Bbbk$. In this case there is no need to introduce monomials t^ρ with exponents $\rho \in \Bbbk$. We define the differential subfield $\mathcal{K}$ of $\mathcal{R}$ as

$$\mathcal{K} := \Bbbk(z)(\!(x)\!).$$

One easily checks that $\mathcal{K}$ is indeed differentially closed with respect to $\partial_{\mathcal{R}}$. Moreover, its field of constants is given by

$$\mathcal{C}_{\mathcal{K}} = \Bbbk(z^p)(\!(x^p)\!).$$

The assumption on the local exponents allows one to modify the normal form theorem to use the function space

$$\mathcal{G}^\rho := x^\rho \sum_{k=0}^{\infty} \bigoplus_{\alpha \in \mathcal{A}_k} \Bbbk z^\alpha x^k,$$

instead of

$$\mathcal{F}^\rho = t^\rho \sum_{k=0}^{\infty} \bigoplus_{\alpha \in \mathcal{A}_k} \Bbbk z^\alpha x^k,$$

by "substituting $t = x$" and analogously one obtains a full basis of solutions over $\mathcal{C}_{\mathcal{K}}$ in $\mathcal{K}$: For each local exponent ρ one computes $u^{-1}(x^\rho)$ instead of $u^{-1}(t^\rho)$, where u is the automorphism described in the normal form theorem.

Polynomial solutions. It is well-known that if a Laurent series solution $y \in \mathbb{F}_p(\!(x)\!)$ to $Ly = 0$ for an operator $L \in \mathbb{F}_p[x][\partial]$ with polynomial coefficients exists, then there already exists a polynomial solution to the equation, see [Hon81] p. 174. We generalize the result to solutions involving only finitely many of the variables z_i.

Lemma 4.4. *Let $\Bbbk$ be a field of characteristic p. Let $L \in \Bbbk[x][\partial]$ be a differential operator with local exponent $\rho \in \Bbbk$. Let $y \in t^\rho \Bbbk[z_1, \ldots, z_\ell][\![x]\!]$ be a solution of the differential equation $Ly = 0$ involving only finitely many of the variables z_i. Let $c \in \mathbb{N}$. Then there exists a polynomial $q \in \Bbbk[x, z_1, \ldots, z_\ell]$, such that $L(t^\rho q) = 0$ and $y - t^\rho q \in t^\rho x^{c+1} \Bbbk[z_1, \ldots, z_\ell][\![x]\!]$. In particular, if a basis of power series solutions of $Ly = 0$ in $\bigoplus_\rho t^\rho \Bbbk[z_1, \ldots, z_\ell][\![x]\!]$ exists, then there already exists a basis of polynomial solutions in $\bigoplus_\rho t^\rho \Bbbk[z_1, \ldots, z_\ell, x]$.*

The proof of Honda can be easily adapted to this generalisation. However, we give a more conceptual proof.

Proof. We consider $t^\rho \Bbbk[z_1, \ldots, z_\ell][\![x]\!]$ as a free $\Bbbk[z_1^p, \ldots, z_\ell^p][\![x^p]\!]$-module of rank $p^{\ell+1}$ with basis $\mathcal{G} = \{t^\rho x^k z^\alpha | k \in \{0, 1, \ldots, p-1\}, \alpha \in \{0, 1, \ldots, p-1\}^\ell\}$. Without loss of generality assume that $\rho = 0$. We can write

$$y(x) = \sum_{g \in \mathcal{G}} y_g(z_1^p, \ldots, z_\ell^p, x^p) g$$

with series $y_g \in \Bbbk[z_1, \ldots, z_\ell][\![x]\!]$. Then

$$Ly = \sum_{g \in \mathcal{G}} y_g(z_1^p, \ldots, z_\ell^p, x^p) L(g) = 0$$

implies that the series $y_g(z_1^p, \ldots, z_\ell^p, x^p)$ form a $\Bbbk[z_1^p, \ldots, z_\ell^p][\![x^p]\!]$-linear relation between the polynomials $L(g)$ in the finite free $\Bbbk[z_1^p, \ldots, z_\ell^p, x^p]$-module $\Bbbk[z_1, \ldots, z_\ell, x]$ for $g \in \mathcal{G}$. By the flatness of $\Bbbk[z_1^p, \ldots, z_\ell^p][\![x^p]\!]$ over $\Bbbk[z_1^p, \ldots, z_\ell^p, x^p]$ there are polynomials $q_g(z_1^p, \ldots, z_\ell^p, x^p) \in \Bbbk[z_1^p, \ldots, z_\ell^p, x^p]$ approximating $y_g(z_1^p, \ldots, z_\ell^p, x^p)$ up to any prescribed degree and such that

$$\sum_{g \in \mathcal{G}} q_g(z_1^p, \ldots, z_\ell^p, x^p) L(g) = 0.$$

Now set

$$q(z_1, \ldots, z_\ell, x) = \sum_{g \in \mathcal{G}} q_g(z_1^p, \ldots, z_\ell^p, x^p) g$$

to get the required polynomial solution of $Ly = 0$. □

Remark 4.5. (i) Assume that $L \in \Bbbk[x][\partial]$, where $\Bbbk$ is a *finite* field of characteristic p with algebraic closure $\overline{\Bbbk}$. Then if $y \in t^\rho \overline{\Bbbk}[\![x]\!]$ is a solution obtained by the normal form theorem, we already have $y \in t^\rho \Bbbk(\rho)[\![x]\!]$, where $\Bbbk(\rho)$ is a finite extension of $\Bbbk$. Recall the operators S and T from the normal form theorem: S is a right inverse to L_0 and $T = L - L_0$. It holds $S(x^{\rho+k+p}) = x^p S(x^{\rho+k})$ and $T(x^{\rho+k+p}) = x^p T(x^{\rho+k})$. There are only finitely many n-tuples of elements from $\Bbbk(\rho)$. Write $y = t^\rho(a_0 + a_1 + a_2 x^2 + \ldots)$. Two n-tuples of consecutive coefficients a_i of y, starting at powers of an index divisible by p, have to agree. Thus the sequence $(a_i)_{i \in \mathbb{N}}$ becomes periodic. Hence it suffices to take a suitable sufficiently large k to obtain a polynomial solution $(1 - x^{kp})y$ of $Ly = 0$, which approximates y to a prescribed degree c.

(ii) The algorithm from the normal form theorem may but need not provide us with a polynomial solution of $Ly = 0$, when applied to an operator L in $\mathbb{F}_p[x][\partial]$. To see this consider the following two examples:

(a) Let $L = x\partial - x^2\partial - x$ and

$$y_L(x) = \frac{1}{1-x}$$

the solution of the equation $Ly = 0$. Over $\mathbb{F}_p$ we compute using the algorithm from the normal form theorem with $L_0 = x\partial$ and $T = x^2\partial + x$ and obtain $u^{-1}(1) = \sum_{k=0}^\infty (S_L \circ T_L)^k = 1 + x + x^2 + \ldots + x^{p-1} \in \mathbb{F}_p[x]$, a polynomial solution.

So we obtain $u^{-1}(1) = \sum_{k=0}^\infty (S_L \circ T_L)^k = 1 + x + x^2 + \ldots + x^{p-1} \in \mathbb{F}_p[x]$, a polynomial solution.

(b) Let now $M = (-x - 2x^4) + (x + x^2 + -2x^4 - x^5 + x^7)\partial$. The equation $My = 0$ is satisfied by the algebraic function $1 + \frac{x}{1-x^3}$. Reducing modulo 3 we get

$$T = (x + 2x^2\partial) + (2x^4\partial) + (2x^4 + x^5\partial) + (2x^7\partial) = T_1 + T_3 + T_4 + T_6$$

and the initial form $M_0 = x\partial$. We compute the solution

$$u^{-1}(1) = \sum_{i=0}^\infty a_i x^i \sum_{i=0}^\infty (S \circ T)^i(1) = 1 + x + x^4 + x^7 + x^{10} + \ldots$$

Because the maximal shift of T is 6 and $(a_1, a_2, a_3, a_4, a_5, a_6) = (a_4, a_5, a_6, a_7, a_8, a_9)$ the sequence of coefficients of this series becomes periodic, as described in (i), with period length 3. Thus, the solution obtained by the normal form theorem in characteristic p agrees with the reduction modulo p of the solution obtained in characteristic 0.

(iii) The latter of the two examples from above illustrates that the degree of a minimal degree polynomial solution of a differential equation in characteristic p need not be $p - 1$, as one could expect. Indeed using the periodicity of the coefficients of the solution from above one obtains that

$$\widehat{y}(x) = u^{-1}(1) - x^3 u^{-1}(1) = 1 + x - x^3$$

is a polynomial solution. Any other polynomial solution has to be a multiple of $\widehat{y}$ with a constant. Indeed, making the ansatz

$$(1 + x - x^3) \cdot (1 + c_1 x^3 + c_2 x^6 + \cdots) = 1 + ax + bx^2$$

one immediately obtains $c_1 = 1$, which leads to a contradiction. Therefore no polynomial solution of degree less than 3 exists.

The p-curvature. Let L be a differential operator. We define the *p-curvature* of L to be the action of multiplication by ∂^p on the space $\mathbb{k}[x][\partial]/\mathbb{k}[x][\partial]L$.

Operators with nilpotent p-curavture. One class of operators with all local exponents in the prime field of $\mathbb{k}$ turn out to be operators with nilpotent p-curvature. An alternative description of these operators was provided by Honda [Hon81] p. 176: We say that an equation $Ly = 0$ of order n has *sufficiently many solutions in the weak sense* if $Ly = 0$ has one solution $y_1 \in \mathbb{k}[\![x]\!]$ and recursively the equation in u' of order $n-1$ obtained from $Ly = 0$ by the ansatz $y = y_1 u$ has sufficiently many solutions in the weak sense.

Theorem 4.6 (Honda, [Hon81], p. 201)**.** *A linear differential operator $L \in \mathbb{k}[x][\partial]$ has nilpotent p-curvature if and only if the equation $Ly = 0$ has sufficiently many solutions in the weak sense.*

Indeed, the following theorem holds:

Theorem 4.7. *Let $L \in \mathbb{k}[x][\partial]$ be a differential operator with nilpotent p-curvature. Then its local exponents are in the prime field $\mathbb{F}_p \subseteq \mathbb{k}$.*

For a proof, see [Hon81] p. 179. Further, there is another interesting characterisation of operators with nilpotent p-curvature due to Dwork [Dwo90]. They are exactly those operators, for which finitely many of the variables z_i suffice to obtain a full basis of solutions:

Theorem 4.8 (Dwork, [Dwo90], p. 756)**.** *An operator $L \in \mathbb{k}[x][\partial]$ has nilpotent p-curvature if and only if there is $l \in \mathbb{N}$ such that $Ly = 0$ has a full basis of solutions in $\mathbb{k}(z_1, \dots, z_l)(\!(x)\!)$ over its field of constants $\mathbb{k}(z_1^p, \dots, z_l^p)(\!(x^p)\!)$.*

This is a generalisation of a result of Honda, who proved the result for $l = 1$ and operators of order smaller than p, see [Hon81] p. 186.

For example, the operator annihilating $\log(1-x)$, discussed in Example 4.3, has nilpotent p-curvature.

Corollary 4.9. *Let $L \in \mathbb{k}[x][\partial]$ be an operator with nilpotent p-curvature. Then there is $\ell \in \mathbb{N}$, such that there is a basis of polynomial solutions of $Ly = 0$ in $\mathbb{k}(x, z_1, \dots, z_\ell)$.*

This immediately follows from Theorem 4.7 and Lemma 4.4.

4.3 The Grothendieck p-curvature conjecture

We now turn to conjectures of Grothendieck-Katz, André, Bézivin, Christol, the Chudnovsky brothers, Matzat and van der Put about the algebraicity of solutions of linear differential equations with polynomial coefficients defined over $\mathbb{Q}$ [Kat70; Kat72; Kat82; And04; Béz91; Chr90; CC83; Mat06; Put96]. The goal is to study them using the normal form theorems in characteristic 0 and p.

It is a classical result, already known to Abel, that algebraic power series satisfy a linear differential equation with polynomial coefficients. The intriguing and meanwhile notorious problem is to characterize those differential equations which arise in this way, a question which appears over and over again in the literature (Abel, Riemann, Autonne, Fuchs, Frobenius, Schwarz, Beukers-Heckman, ...).

In the previous section we have studied operators with nilpotent p-curvature. We want to study now operators L with vanishing p-curvature, i.e., L divides ∂^p from the right. The vanishing of the p-curvature of an operator can be described in terms of its solutions:

Lemma 4.10 (Cartier)**.** *Let $L \in \mathbb{k}[x][\partial]$ be a differential operator, where $\mathbb{k}$ denotes a field of characteristic p. Then L admits a full basis of solutions over $\mathbb{k}(\!(x^p)\!)$ in $\mathbb{k}[x]$ if and only if the p-curvature of L vanishes.*

The original abstract formulation and a proof can be found in [Kat70], a more "down-to-earth" proof in [SP03]. Compare this result also to Corollary 4.9.

In the following let $L \in \mathbb{Q}[x][\partial]$ be a differential operator defined over $\mathbb{Q}$ and denote by $L_p \in \mathbb{F}_p[x][\partial]$ the differential operator that arises from reducing the coefficients of L modulo p, whenever this is defined. The reduction L_p is defined for all but finitely many prime numbers p. We are interested in the interplay between solutions of the equations $Ly = 0$ and $L_p y = 0$. Most prominent here is the Grothendieck p-curvature conjecture.

We now give an elementary formulation of the Grothendieck p-curvature conjecture. In this formulation the p-curvature does not appear, however Cartier's Lemma 4.10 establishes the connection.

Conjecture 4.11 (Grothendieck p-curvature conjecture, [Hon81])**.** *Let $L \in \mathbb{Q}[x][\partial]$. Assume that $L_p y = 0$ has a basis of $\mathbb{F}_p[\![x^p]\!]$-linearly independent solutions in $\mathbb{F}_p[\![x]\!]$ for almost all prime numbers p. Then there exists a basis of $\mathbb{Q}$-linearly independent algebraic solutions of $Ly = 0$ in $\mathbb{Q}[\![x]\!]$.*

Remark 4.12. One can easily generalize this conjecture to number fields, by replacing $\mathbb{Q}$ with $K = \mathbb{Q}(\alpha)$, for α an algebraic number, and $\mathbb{F}_p$ by the residue fields of $\mathcal{O}_K$ modulo its prime ideals $\mathfrak{p}$.

The case of order one equations is equivalent to a special case of a theorem of Kronecker (which, in turn, is a special case of Chebotarev's density theorem) [Hon81]. Katz has proven the conjecture for Picard-Fuchs equations [Kat72]. There have been recent and quite technical advances in the conjecture by various people, but the general case (even for order two equations) seems to still resist. Bost has established a more general variant of the conjecture for algebraic foliations and subgroups of Lie-groups [Bos01], [Cha02], Thm. 2.4. Progress was also made by Farb and Kisin[FK09] as well as Calegari, Dimitrov and Tang [CDT21].

An apparently weaker statement than the Grothendieck conjecture was proposed by Bézivin.

Conjecture 4.13 (Bézivin conjecture, [Béz91])**.** *Let $L \in \mathbb{Q}[x][\partial]$ be a differential operator. Assume that $Ly = 0$ has a basis of $\mathbb{Q}$-linearly independent solutions in $\mathbb{Z}[\![x]\!]$. Then these solutions are algebraic over $\mathbb{Q}(x)$.*

Lemma 4.14. *The validity of the Grothendieck p-curvature conjecture implies the validity of the Bézivin conjecture.*

In other words: The hypothesis of the Bézivin conjecture implies the hypothesis of the Grothendieck p-curvature conjecture.

Proof. Assume that $y \in \mathbb{Z}[\![x]\!]$ is an integral solution of $Ly = 0$. Its reduction modulo all prime numbers is well-defined and a solution to $L_p y = 0$. For p larger than the maximal difference of the local exponents of L, a basis of solutions of $Ly = 0$ gets mapped by reduction to a basis of solution modulo p. The condition on p is necessary to ensure that the reductions of the solutions do not become linearly dependent over $\mathbb{F}_p(\!(x^p)\!)$. Thus by the Grothendieck p-curvature conjecture $Ly = 0$ has a basis of algebraic solutions and y, as a linear combination of those algebraic solutions, is algebraic itself. $\square$

A substantial advance towards the Grothendieck p-curvature conjecture would be to prove the inverse implication of Lemma 4.14: in fact it would transfer the problem from positive characteristic to characteristic 0. To approach the converse implication, it is reasonable to compare the algorithm of the normal form theorem in characteristic 0 applied to an operator L to the algorithm of the normal form theorem in characteristic p, applied to the reduction L_p of the operator L modulo p. We investigate in the next paragraphs how the normal form theorems could be used to achieve this.

The problem which arises lies in the observation that the characteristic p algorithm does not entirely coincide with the reduction modulo p of the algorithm in zero characteristic. Very subtle disparities appear, and this makes it hard to deduce properties of the characteristic zero solutions from the characteristic p solutions, in particular, to prove their algebraicity. One hope is, however, to be able to compare the Grothendieck-Katz conjecture with the Bézivin conjecture.

We will use the following number theoretic result:

Theorem 4.15 (Kronecker, [Kro80], Frobenius, [Fro96]). *Let $f \in \mathbb{Q}[x]$ be a polynomial of degree n, let $s \in \mathbb{N}$ and $n_1, \ldots, n_s$ with $n_1 + \ldots + n_s = n$. The density of prime numbers p for which the reduction of f modulo p splits into k factors of degrees $f_1, \ldots, f_k$ is equal to the number of permutations of the roots of f in the Galois group of f consisting of s cycles of lengths $f_1, \ldots, f_s$. In particular, f splits into linear factors over $\mathbb{Q}[x]$ if and only if its reduction modulo p splits in $\mathbb{F}_p[x]$ into linear factors for almost all primes p.*

This version was proven by Frobenius, while similar results were formulated by Kronecker before. It is also an easy corollary of the Chebotarev density theorem.

We now describe consequences of the hypothesis of the Grothendieck p-curvature conjecture. They were already collected by Honda and we refer for parts of the proof to his article. However, for the last assertion we give a different proof. It compares the two algorithms obtained from the normal form theorems in characteristics 0 and p.

For an operator $L \in \mathbb{Q}[x][\partial]$ in characteristic 0 and a fixed prime p we denote in the sequel by $\overline{L} = L_p \in \mathbb{F}_p[x][\partial]$ the reduction of L modulo p, whenever this reduction is defined.

Proposition 4.16 (Honda). *Let $L \in \mathbb{Q}[x][\partial]$ be a differential operator with polynomial coefficients over $\mathbb{Q}$. Assume that the induced equations $\overline{L}y = 0$ modulo p have an $\mathbb{F}_p[\![x^p]\!]$-basis of power series solutions in $\mathbb{F}_p[\![x]\!]$, for almost all primes p. Then*

(a) The operator L has a regular singularity at 0.

(b) The local exponents of L at 0 are pairwise distinct rational numbers $\rho_i \in \mathbb{Q}$.

(c) There exists a $\mathbb{Q}$-basis of Puiseux series solutions $y(x)$ of $Ly = 0$ in $\sum_{\rho_i} x^{\rho_i}\mathbb{Q}[\![x]\!]$, where ρ_i ranges over the local exponents of L. In particular, this basis is independent of the variables t and z_i in $\mathcal{R} = \bigoplus_{\rho \in \mathbb{k}} t^\rho \mathbb{k}(z_1, z_2, \ldots)(\!(x)\!)$.

Proof. For part (a) use [Hon81], Corollary p. 178, combined with Theorem. 4.6 and Lemma 4.10 from above.

(b) See [Hon81], Thm. 2, p. 179, combined with Thms. 4.7 and 4.15 from above. We provide here a variant of Honda's proof. As a consequence of (a) there are n local exponents of L, counted with multiplicity, $n = \operatorname{ord} L$. Moreover, for almost all primes p, the local exponents of $\overline{L}$ have to be elements of the prime field $\mathbb{F}_p$. Indeed for any local exponent $\rho \notin \mathbb{F}_p$ we obtain using Theorem 3.17 a solution of the form $t^\rho f(x) \in t^\rho \mathbb{F}[\![x]\!]$, contradicting the existence of a basis of n solutions of $\overline{L}y = 0$ in $\mathbb{F}_p[\![x]\!]$.

It is then shown as in [Hon81] that the local exponents of L are pairwise incongruent modulo almost all primes p. The indicial polynomial χ_L of L has coefficients in $\mathbb{Q}$ and its reduction modulo p splits into linear factors over $\mathbb{F}_p$ for almost all primes p. Thus, by Theorem 4.15, χ_L splits into linear factors over $\mathbb{Q}$. It follows that all local exponents of L are rational. Assume now that two local exponents are congruent modulo some p. Then their reduction modulo p is a local exponent of $\overline{L}$ of multiplicity at least 2. So, Theorem 3.17 together with the remarks in section 4.2 upon avoiding the variable t, yield a solution of $\overline{L}y = 0$ of the form $u^{-1}(x^\rho z_1)$, where u is the automorphism of the normal form theorem in positive characteristic, Theorem 3.16. This solution now depends on z_1, contradictory to the assumption. This proves (b).

(c) By Theorem 2.16 a basis of solutions of $Ly = 0$ lies in

$$\sum_{\rho_i} x^{\rho_i}\mathbb{Q}[\![x]\!][z],$$

the sum varying over all local exponents ρ_i of L. It remains to prove that these solutions are independent of z. So assume the contrary: let f be a solution which depends on z. Without loss of generality we may assume that

$$f = u^{-1}(x^\rho) = x^\rho(1 + a_1(z)x + a_2(z)x^2 + \ldots)$$

for some local exponent $\rho \in \mathbb{Q}$ of L and some $a_i \in \mathbb{Q}[z]$. Let $m \in \mathbb{N}$ be the first index where a_m depends on z. We will construct from f a solution g of $\overline{L}y = 0$, for a suitable prime p, which involves z_1-terms which are not p-th powers. This will produce the required contradiction.

The construction of g is, in fact, quite subtle. We have to run the two normal form algorithms for the construction of f and g simultaneously in characteristic 0 and p as long as no z appears in characteristic 0. At the moment when z occurs for the first time, say, in the computation of the coefficient a_m of f, a careful comparison ensures that z_1 shows up also in the expansion of g in the characteristic p algorithm.

We choose the prime p subject to the following conditions:

- $p > n = \operatorname{ord} L$;
- There is a basis of solutions of $\overline{L}y = 0$ in $\mathbb{F}_p[\![x]\!]$;
- p does not divide any of the denominators of the local exponents of L;
- p does not divide any of the denominators of the coefficients of $a_1, \ldots, a_k$.

Let Λ be the set of positive integers ℓ smaller than m such that $\sigma := \rho + \ell$ is a local exponent of $\overline{L}$. Here we write σ and ρ for elements in $\mathbb{Q}$ as well as for the representatives in $\{0, 1, \ldots, p-1\}$ of their reduction modulo p. We define

$$g = \overline{u}^{-1}\left(x^\rho\left(1 + \sum_{\ell \in \Lambda} \overline{a_\ell} x^\ell\right)\right) = x^\rho(1 + b_1(t,z)x + b_2(t,z)x^2 + \ldots),$$

where $\overline{u}$ is the automorphism of $\mathcal{G}^\rho = x^\rho \sum_{k=0}^\infty \bigoplus_{\alpha \in \mathcal{A}_k} \Bbbk z^\alpha x^k$ from the normal form theorem, Theorem 3.16, compare again to the remarks on avoiding the variable t in Section 4.2. The additional summand $\sum_{\ell \in \Lambda} \overline{a_\ell} x^\ell$ in the inner parenthesis of g is required to make f and g coincide up to degree $m - 1$.

We will show that g is a solution of $\overline{L}y = 0$ and that its coefficient b_m involves z_1. The first thing is easy since, by the normal form theorem 3.16,

$$\overline{L}(g) = (\overline{L}_0 \circ \overline{u})(g) = \overline{L}_0\left(x^\rho\left(1 + \sum_{\ell \in \Lambda} \overline{a_\ell} x^\ell\right)\right) = 0,$$

for $\rho + \ell$ is a local exponent of $\overline{L}_0$ and hence $\overline{L}_0(x^{\rho+\ell}) = 0$. This proves $\overline{L}g = 0$.

Next we prove inductively that $b_\ell = \overline{a_\ell}$ for $\ell \leq m - 1$, i.e., that the expansion of g up to degree $m - 1$ equals the reduction of the respective expansion of f. This part is a bit computational.

Write $T = L - L_0$ and $\overline{T} = \overline{L} - \overline{L}_0$ as earlier for the tails of L and $\overline{L}$. We expand T and $\overline{T}$ as sums of Euler operators

$$T = T_1 + \cdots + T_r,$$
$$\overline{T} = \overline{T}_1 + \cdots + \overline{T}_r,$$

Similarly, we define S and $\overline{S}$ as the inverses $S = (L_0|_{\mathcal{H}})^{-1}$ and $\overline{S} = (\overline{L}_0|_{\overline{\mathcal{H}}})^{-1}$ of L_0 and $\overline{L}_0$, respectively, on direct complements of their kernels, as described in the normal form theorems, Theorems 2.15 and 3.16.

We now distinguish two cases.

(i) Assume first that $\rho+\ell$ is not a local exponent of L. Rewriting the differential equations $Ly = 0$ and $\overline{L}y = 0$ as linear recursions for the coefficients of the prospective solutions we obtain, for $\ell \leq m-1$,

$$a_\ell = S\left(\sum_{k=1}^{r} T_k\left(a_{\ell-k}x^{\rho+\ell-k}\right)\right),$$

and

$$b_\ell = \overline{S}\left(\sum_{k=1}^{r} \overline{T}_k\left(b_{\ell-k}x^{\rho+\ell-k}\right)\right),$$

where both sums in the parentheses are homogeneous of degree $\rho+\ell$ in x. By induction on ℓ we may assume that $b_{\ell-k} = \overline{a_{\ell-k}}$ equals the reduction of $a_{\ell-k}$ for all $k = 1,\ldots,r$. Hence this also holds for $b_\ell = \overline{a_\ell}$.

(ii) Assume now that $\rho+\ell$ is a local exponent of L. Here, the formula for b_ℓ is different, by the very definition of g,

$$b_\ell = \overline{S}\left(\sum_{k=1}^{r} \overline{T}_k(b_{\ell-k}x^{\rho+\ell-k})\right) + \overline{a_\ell}.$$

Now, as $\rho+\ell$ is a local exponent of L and hence also of $\overline{L}$, the image $\overline{S}(x^{\rho+\ell})$ will involve z_1. Therefore, as b_ℓ does not involve z_1 by assumption, we get $\overline{T}_k(b_{\ell-k}x^{\rho+\ell-k}) = 0$. Hence again $b_\ell = \overline{a_\ell}$ for all $\ell \leq m-1$.

This proves in both cases that g is the reduction of f modulo p up to degree $m-1$.

To finish the proof we will show that b_m involves z_1. This will produce the required contradiction. As a_m depends on z by assumption, $\rho+m$ is necessarily a local exponent of L and thus of $\overline{L}$. Hence $\overline{S}(x^{\rho+m})$ will depend on z_1. Recall that

$$a_m = S\left(\sum_{k=1}^{r} T_k(a_{m-k}x^{\rho+m-k})\right),$$

and

$$b_m = \overline{S}\left(\sum_{k=1}^{r} \overline{T}_k(b_{m-k}x^{\rho+m-k})\right).$$

From the already established equalities $b_\ell = \overline{a_\ell}$ for $\ell \leq m-1$ it follows that $\overline{T}_k(b_{m-k}x^{\rho+m-k})$ is the reduction modulo p of $T_k(a_{m-k}x^{\rho+m-k})$. Now, if $T_k(a_{m-k}x^{\rho+m-k})$ were 0, its image a_m under S were zero, which is excluded by the choice of m. So this term is non-zero. But then it suffices to choose p sufficiently large such that also the reduction $\overline{T}_k(b_{m-k}x^{\rho+m-k})$ is non-zero. Similarly as before we then get that b_m involves z_1, contradiction. $\square$

We illustrate the crucial step in the proof of (c) by an example.

Example 4.17. The operator $L = x^2\partial^2 - 3x\partial - 3x - x^2 - x^3$ has the solution

$$f(x) = u^{-1}(1) = 1 + a_1x + a_2x^2 + \ldots = 1 - x + \frac{1}{2}x^2 - \frac{1}{2}x^3 - \frac{1}{2}x^4z + \ldots,$$

so $a_4 = -\frac{1}{2}z$ is the first coefficient which depends on z. Assume that there was a full basis of solutions in $\mathbb{F}_3[\![x]\!]$. The local exponents in characteristic 3 are 0 and 1, so $\Omega_4 = \{1,3\}$. We compute, using $\overline{T} = x^2 + x^3$ and $\overline{L}_0 = x^2\partial^2$, the expansion of the following solution

$$\overline{u}_3^{-1}(1 + 2x + x^3) = 1 + 2x + 2x^2 + x^3 + \ldots,$$

which agrees with the reduction of f up to order 3. However, the next term in the expansion is $S_3(x^4) = x^4z_1$, so $u_3^{-1}(1 + 2x + x^3) \notin \mathbb{F}_3[\![x]\!]$.

4.4 Outlook

If one wants to pursue the goal of proving the equivalence of the Grothendieck p-curvature conjecture and the Bézivin conjecture, number theoretic obstacles occur.

A power series $y(x) \in \mathbb{Q}[\![x]\!]$ is called *globally bounded* if there is an integer N such that $y(Nx) \in \mathbb{Z}[\![x]\!]$. In other words, there are only finitely many prime numbers p appearing in the denominators of the coefficients of y and they only grow geometrically. A theorem of Eisenstein [Eis52] says that any algebraic power series is globally bounded.

To prove that the validity of the Bézivin conjecture implies the validity of the Grothendieck p-curvature conjecture it suffices to show that for a linear differential equation $Ly = 0$ whose reduction $L_p y = 0$ has a full basis of solutions in $\mathbb{F}_p[\![x]\!]$ the basis of solutions in characteristic 0 is globally bounded. For this it is natural to try to compare the algorithms from the normal form theorems in characteristic 0 and p further. Ideally, p would not appear in the denominators of solutions in characteristic p if and only if there is a basis of solutions in $\mathbb{F}_p[\![x]\!]$ of $L_p y = 0$, at least for almost all p. However, the situation is not as easy as one might hope, as the following two examples illustrate:

Example 4.18. (i) The first example shows that for finitely many primes it may happen that a full basis of solutions of the reduction of a linear differential equation modulo p exists, although p appears in the denominator of one of the solutions in characteristic 0. The solution of $\partial - nx^{n-1}$ for $n \in \mathbb{N}$ is e^{x^n}, a power series where each prime number appears eventually in the denominators. However, for all prime numbers p dividing n, the reduction of the equation modulo p is an Euler equation having the solution $1 \in \mathbb{F}_p[\![x]\!]$. As this can happen only for a finite number of primes, this does not contradict the Grothendieck p-curvature conjecture.

(ii) The next example shows that to rule out the appearance of the prime factor p in the denominators of a solution of $Ly = 0$ it is not sufficient to work on the level of individual solutions associated to a local exponent and its reduction. If possible at all, it has to take into account the existence of a full basis of solutions.

The power series

$$y(x) = \sum_{k=1}^{\infty} a_k x^k = \sum_{k=1}^{\infty} \frac{k(k+2)}{(k+1)} x^k = \frac{3}{2}x + \frac{8}{3}x^2 + \frac{15}{4}x^3 + \frac{24}{5}x^4 + \ldots = \frac{\log(1-x)}{x} + \frac{x}{(x-1)^2}$$

is annihilated by the third order operator

$$L = x^3\partial^3 + 4x^2\partial^2 + x\partial - 1 - (x^4\partial^3 + 8x^3\partial^2 + 13x^2\partial + 3x).$$

This operator L is *hypergeometric*, i.e., $T = L - L_0$ is an Euler operator with shift one. Moreover, y is annihilated by the second order operator

$$M = 3x^2\partial^2 + 3x\partial - 3 + (x^4 - 4x^3)\partial^2 + (3x - 12x^2)\partial + x^2 - 4x,$$

which is not hypergeometric. The operator M is a right divisor of L, as one verifies that

$$\left(-\frac{1}{x-3}x\partial - \frac{1}{x-3}\right)M = L.$$

Let us first concern ourselves with the operator L. Its local exponents are -1 with multiplicity two and 1 with multiplicity 1. We have $y = \frac{3}{2} \cdot u^{-1}(x)$, where u is the automorphism described in the normal form theorem in characteristic 0. Moreover we compute $u^{-1}(x^{-1}) = x^{-1}$ and $u^{-1}(x^{-1}z) = x^{-1}z$. Thus a basis of solutions of $Ly = 0$ is given by y, x^{-1} and $x^{-1}\log(x)$.

For all prime numbers p the coefficient of x^{p-2} in the expansion of y is divisible by p, while the denominators of $a_1, \ldots, a_{p-2}$ are not. Thus

$$y_p := \sum_{k=1}^{p-2} a_k x^k$$

is well defined in characteristic p and a solution to the equation $L_p y = 0$. It is given as $u_p^{-1}(x)$ where u_p is the automorphism defined in the normal form theorem in characteristic p. The series y is not algebraic, as it is not globally bounded. In fact any prime number p appears in the denominators of the coefficients a_i. However, the solution in characteristic p corresponding to the reduction of the local exponent 1 is a genuine power series. Other linearly independent solutions in characteristic p are x^{-1} and $x^{-1}z_1$. We see that in neither characteristic there is a basis of power series solutions.

Let us now turn to the operator M, which has local exponents -1 and 1 as well, both with multiplicity 1. A basis of solutions is given by x^{-1} and y. This does not contradict the Grothendieck p-curvature conjecture, as y_p is not a solution of M. For L the construction was very dependent on the fact that the equation is hypergeometric, which is no longer the case for M.

There still remain several questions about linear differential equations over fields with positive characteristic. For linear differential equations with holomorphic coefficients there is a criterion by Fuchs characterizing regular singular points of an operator L [Fuc66]. A point $a \in \mathbb{P}^1_{\mathbb{C}}$ is at most a regular singularity of L if and only if there is a local basis of solutions of $Ly = 0$, which grows at most polynomially when approaching a. One would expect a similar criterion in characteristic p: an n-dimensional vector space of solutions in $\mathcal{R}$ over the constants $\mathcal{C}$ should suffice to conclude that 0 is a regular singular point of L. The needed framework could be provided by adapting the solution theory of linear differential equations with holomorphic coefficients and an irregular singularitiy at 0 of N. Merkl, described in section 2.4 to positive characteristic. More precisely, this should allow the definition of a ring $\widetilde{\mathcal{R}}$ in which every linear differential equations with an irregular singularity at 0 in positive characteristic has a basis of solutions. The corresponding criterion in characteristic p should then read: A linear differential equations $Ly = 0$ admits a basis of solutions in $\mathcal{R} \subseteq \widetilde{\mathcal{R}}$ if and only if 0 is a regular singularity of L

Moreover, the solutions of differential equations in $\mathcal{R}$ need to be better understood. For example one would expect some kind of pattern in the exponential function in positive characteristic discussed in Example 4.1. However, no such structure seems obvious. Also the question of the algebraicity of the constant term raised in Example 4.1 and Problem 4.2 deserves some attention and should be studied for the constant terms of solutions of any equation.

In addition there is hope to extract information about the p-curvature of linear differential operator L in positive characteristics from the description of a full basis of solutions in the differential extension $\mathcal{R}$ of $\mathbb{k}$. In [BCS15] Bostan, Caruso and Schost describe an algorithm on how to effectively compute the p-curvature of a differential operator. They work over the ring $\mathbb{k}[\![x]\!]^{\mathrm{dp}}$ of series of the form

$$f = a_0 + a_1\gamma_1(x) + a_2\gamma_2(x) + \dots .$$

The elements $\gamma_i(x)$ are formal variables, but should be thought of $\frac{x^i}{i!}$. The multiplication on $\mathbb{k}[\![x]\!]^{\mathrm{dp}}$ is consequently given by $\gamma_i(x)\gamma_j(x) = \binom{i+j}{i}\gamma_{i+j}(x)$. In a suitable extension extension of $\mathbb{k}[\![x]\!]^{\mathrm{dp}}$, accounting for local exponents outside the prime field, they construct a basis of solutions of $Ly = 0$. From this basis they compute, passing to systems of first order equations, the matrix representation of the p-curvature. A similar program seems feasible working in $\mathcal{R}$ instead of $\mathbb{k}[\![x]\!]^{\mathrm{dp}}$.

Finally, there remain, of course, the Grothendieck p-curvature conjecture and the Bézivin conjecture. As Example 4.18 shows, the algorithms of the normal form theorems in characteristic p and 0 show some unexpected discrepancy. The hope that solutions of the reduction of differential operators are reductions of solutions of the operator seems to be unfounded. However, the phenomena shown require further investigation.

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University of Vienna, Faculty of Mathematics, Oskar-Morgenstern-Platz 1, 1090, Vienna, Austria
Email: `florian.fuernsinn@univie.ac.at`
Email: `herwig.hauser@univie.ac.at`